

Metodi perturbativi

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Chapter 1

Introduction

Perturbative methods are always applied to mathematical models; when a model is very complicated, it is possible to exploit the fact that there is some very little quantity with respect to the other ones, in order to modify the model, transforming it in an easier model. Therefore, the most significant point is the determination of something “little”; what does “little” means? It is necessary to determine some criteria aimed at studying quantities.

When complicated models should be used, even the straight application of numerical methods may be not possible or not convenient; if problems are nonlinear or have some instabilities, this procedure can not be used; therefore, also physical models can be useless, if they are too complicated.

As we suggested, the idea is to simplify a model, building something easier starting from it; in some sense, we *distorce* the physics of the problem, introducing some approximations; if our scenario fits in the range of validity of the approximations, it is possible to do this, and to obtain very accurate results; however, a discussion of the validity of this kind of approach should be carried out.

This is the idea of perturbation methods: we look for something small in the problem, then we introduce some modification, approximation to the model, in order to tackle the easier problem by means of numerical methods. An example is the transport model: it is a very complicated model, even if very good, so the direct application of numerical methods on it is not possible; by introducing approximations, it is possible to obtain for instance a diffusion model, which can be studied very easily. Obviously, the transport model degenerates into a diffusion model only in a very specific range of situations; out of these, it is not possible to do this.

Another very important concept is the one of **local solution**: the starting model is able to give information on each point; however, probably, this is too much: many times it is not necessary to obtain all this information, since only some parameters of the phenomena are interesting. Perturbation methods can provide the user with global solutions; these are parameters containing information on the global behavior of the phenomenon, synthesizing information in some numbers. An example may be the solution of transport equation; the solution of transport equation is n , which is the density of neutrons; it is function of several variables:

$$n = n(\underline{r}, E, \underline{\Omega}, t)$$

where \underline{r} is the position, E is energy, $\underline{\Omega}$ is the considered direction, t is time; this is too much information! Many times, we want to handle global information, in order to synthesize it.

Chapter 2

Setting basic perturbation methods

2.1 Introduction: Frobenius vs perturbation methods

As we have discussed, in order to formulate a perturbation method on some problem, it is necessary to identify a parameter which is very small with respect to the other ones; then, we call it ε . We have to recall the fact that our objective is to obtain **easy** models, therefore it is necessary to formulate all this stuff keeping into account of the difficulties of the problem: if ε is related to some part of the problem which creates complications, it is better! For instance, if ε is a non-linear term (which disappears, if we set $\varepsilon = 0$), the problem is well formulated! In other cases, ε may have a physical meaning.

Perturbation methods are based on the following idea: we write the solution of our problem as a linear combination of ε powers; this is a **perturbation expansion**. In order to clarify, let us consider the following differential equation:

$$y'' = \left(1 + \frac{\varepsilon}{1+x^2}\right) y$$

if $\varepsilon = 0$, this problem becomes very easy; in fact, if $\varepsilon = 0$, we obtain the following differential equation:

$$y_0'' = y_0$$

(the 0 indexes that we consider the 0-order approximation). In order to obtain a higher-order approximation, we have to write $y(x)$ as:

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$$

We hope that, knowing y_0 , we can find y_1 , then y_2 , and so on.

It is very important to remark that in this case we are not expanding $y(x)$ as a power series of x , but as a power series of ε ; when we expand as a power series of the independent variable of the problem, we apply the **Frobenius method**. The Frobenius method returns a solution which is very good in the neighborhood of a chosen point (for instance if we expand as $\sum_n (x - x_0)^n$, the approximation will be very good in the neighborhood of $x = x_0$); on the other hand, since the expansion is in ε , what we obtain from a perturbation method is a quite-good **global** approximation; this is completely different from the Frobenius approach, which returns something good only in one point! However, these are different methods for different purposes.

2.1.1 Application to an algebraic equation

As first example, it is possible to focus on an algebraic problem:

$$x^3 - 4.001x + 0.002 = 0$$

we want to solve this problem by means of perturbation methods; it is possible to observe that the coefficient of x equals 4, plus a little bit; let us define ε as that “little bit”:

$$\varepsilon = 0.001$$

so, the algebraic problem can be rewritten as follows:

$$x^3 - (4 + \varepsilon)x + 2\varepsilon = 0$$

ε is small with respect to the other terms; therefore, roots will change as we change ε , but “not too much”, if ε stays in the order of magnitude of 0.001. Since we identified ε , the second point of the procedure is to write the solution of the equation as a perturbation series:

$$x(\varepsilon) = \sum_{n=0}^{N_\varepsilon} a_n \varepsilon^n$$

Let us try to answer to this question: is the problem easier for $\varepsilon = 0$? Well, let us rewrite the equation in this case:

$$x^4 - 4x = 0$$

somehow, it is! Indeed, this equation has three known solutions, which are:

$$x(x^2 - 4) = 0 \implies x = 0, x = +2, x = -2$$

Now, let us consider the root $x_1 = -2$, and let's just focus on in for some time; if $\varepsilon = 0$, it means that we are truncating the series at the zeroth order; then, let us try to find, for this solution, a higher order approximation, e.g. first order; the perturbation series is:

$$x_1(\varepsilon) = a_0 + a_1\varepsilon$$

if $\varepsilon = 0$, $x_1 = a_0$, so x_1 is the root of the problem approximated with the zero order approximation:

$$a_0 = x_1^{(\text{zero-order-approx})} = -2$$

so:

$$x_1 \sim -2 + a_1\varepsilon$$

how to find a_1 ? Well, we still know the algebraic equation, the starting problem, which was the equation

$$x^3 - (4 + \varepsilon)x + 2\varepsilon = 0$$

to find a_1 , it is simply possible to force this approximation to be the solution of this equation; therefore, it is necessary to substitute x_1 in x :

$$(-2 + a_1\varepsilon)^3 - (4 + \varepsilon)x + 2\varepsilon = 0$$

now we are trying to find a 1-order approximation; indeed, $x_1 = a_0 + a_1\varepsilon$; this means that whatever belongs to an order greater than one should be

neglected; so, the terms in $\varepsilon^2, \varepsilon^3$ and so on should be neglected; so, using this procedure:

$$-8 + 12a_1\varepsilon + 8 - 4a_1\varepsilon + 2\varepsilon + 2\varepsilon = 0$$

which is:

$$a_1 = -\frac{1}{2}$$

a remark: the dependence in ε should disappear from the calculations, if everything is correct!

What have we found? Well: to find a_1 , we needed only information about a_0 , so about the approximation of lower order; we didn't use any other information.

It would be nice if a_1 does not change increasing the order of the approximation; this would be similar to the behavior of the eigenfunction expansion of an operator; indeed, if the operator is self-adjoint (and compact), the eigenvectors (eigenfunction, eigenstates) are orthonormal, and so since the expansion functions are the eigenvector, the expansion coefficients are the eigenvalues, an eigenvalue is a function of its eigenvector only: indeed, if they are orthonormal, they give no contribution to the other expansion coefficients.

In this case it is possible to prove that, for the perturbation series, there is the same effect of the eigenfunction expansion: the addition of new terms does not affect the previous coefficients.

So, let us go on:

$$x_1(\varepsilon) = a_0 + a_1\varepsilon + \sum_{n=2}^{\infty} a_n\varepsilon^n$$

this series is $O(\varepsilon^2)$; this means that the error in approximating the expression is $O(\varepsilon^2)$ (in analogy with the notation used for the Taylor expansion, where the error is evaluated on x ; in this case, this is not a local approximation such as the Taylor expansion, but it is global; the error has to be evaluated with respect to ε).

Let us show that adding new terms, we do not have to modify the previous ones; let us consider:

$$x_1(\varepsilon) = -2 + a_1\varepsilon + a_2\varepsilon^2$$

What about the other coefficients? Let us assume to have substituted everything; in this case, we have:

$$() + (I)\varepsilon + (II)\varepsilon^2 = 0$$

just like in the previous case, adding the ε^2 terms. Performing the very same calculations, excluding the terms with order greater than 2, it is possible to find:

$$\begin{cases} a_0 = -2 \\ 8a_1 + 4 = 0 \\ 8a_2 - 6a_1^2 - a_1 = 0 \end{cases}$$

the first equation is trivial; the second one returns again $a_1 = -\frac{1}{2}$; the second one returns (it is possible to prove this):

$$a_2 = \frac{1}{8}$$

So, the second order approximation is:

$$x_1(\varepsilon) = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$$

Now, ε was a term identified before; if we substitute its original expression, which was $\varepsilon = 0.001$, and we compare with the exact solution, we can see if the second order approximation is satisfactory.

All these calculations concern the solution x_1 only! For the solution $x_2 = 0$ (well, this is only $a_0 = 0$, so the zero order approximation only! This is just to identify it!), it is necessary to re-do similar calculations, and a different perturbation series, which means different coefficients a_i , will be found.

A particular case is, for this equation, the solution relative to $x_3 = +2$; indeed, by calculating again the perturbation analysis, it is possible to find that:

$$a_0 = +2 \quad a_1 = 0; \quad a_2 = 0; \quad a_i = 0 \dots$$

so, only a_0 survives; this means that, in this equation and for that particular solution, the perturbation series returns the exact solution (at the zero order approximation!); therefore, it is possible to “capture” the exact behavior of the physical system just by performing this analysis.

2.1.2 Application to a differential equation

Let us consider a more complicated problem; a second order ordinary differential equation of the form:

$$y''(x) = f(x)y(x)$$

for special forms of $f(x)$ these equations exhibit closed form solutions; however, this is not true in every case.

Since what causes troubles in this equation is $f(x)$, we may try to apply ε in front of it; by this way, if $\varepsilon = 0$, the equation becomes much simpler!

Obviously, a differential equation has to be defined in a domain; this domain is defined by the boundary conditions of the equation; let us choose, for this example:

$$y'(0) = 1; y(0) = 1$$

Let us write the perturbative series of this problem:

$$y(x) = \sum_{n=0}^{\infty} y_n(x)\varepsilon^n$$

$y_n(x)$ are functions, but actually in this case they are the **expansion coefficients of the perturbation series**. Just like in the algebraic equation, $y_0(x)$ is the solution relative to the zero order approximation, $y_1(x)$ the solution relative to the first order approximation (only the ε^1 terms), then $y_2(x)$ is relative to ε^2 , and so on. Just like before, it is necessary to truncate the expansion to a certain term, substitute it in the original equation, equate the coefficients of ε^0 (constant), of ε , of ε^2 to zero (in order to satisfy the left hand-side term equal to zero in an equation in ε , every coefficient should be set equal to zero), and then the system is determined.

One slightly different way to proceed is to set $y_0(0) = 1$, $y'_0(0) = 1$; by this way, all the initial conditions are fulfilled with the zero-order terms; let us say that we are “including” the boundary conditions in the expansion. Then, $\forall n > 0$, $y_n(0) = 0$, $y'_n(0) = 0$: indeed, the boundary condition is included in y_0 and in y'_0 , so it should not be introduced in the remaining terms.

Which is the zero order approximation? Well, let us set $\varepsilon = 0$; the differential equation becomes:

$$y''(x) = 0$$

its solution is the straight line:

$$y_0(x) = cx + d$$

since:

$$y'(0) = 1 \implies c = 1$$

and:

$$y_0(0) = 1 \implies d = 1$$

so:

$$y_0(x) = x + 1$$

Now, it is possible to write the perturbation series representation of the solution $y(x)$ as:

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x) = y_0(x) + \sum_{n=1}^{\infty} \varepsilon^n y_n(x) = x + 1 + \sum_{n=1}^{\infty} \varepsilon^n y_n(x)$$

Let us calculate the first and the second derivatives of this expression:

$$y'(x) = 1 + \sum_{n=1}^{\infty} \varepsilon^n y_n'(x)$$

$$y''(x) = \sum_{n=1}^{\infty} \varepsilon^n y_n''(x)$$

so, let us substitute this in the differential equation:

$$\sum_{n=1}^{\infty} \varepsilon^n y_n''(x) = \varepsilon f(x) \left[x + 1 + \sum_{n=1}^{\infty} \varepsilon^n y_n(x) \right]$$

this can be written in an expanded, explicit way:

$$\varepsilon y_1'' + \varepsilon^2 y_2'' + \dots = \varepsilon f(x) [x + 1] + \varepsilon^2 f(x) y_1(x) + \dots$$

so, it is possible to write the following chain of equations:

$$y_1'' = f(x)(x + 1) = f(x)y_0(x)$$

then, it is possible to find the solution plugging in here some approximation:

$$y_2'' = f(x)y_1(x)$$

so, this means that we can use as a source of the problem the solution of the previous step; therefore, in general:

$$y_n'' = f(x)y_{n-1}(x)$$

This approach has been presented on differential equations, but it is extensively used in integral equations; let $\hat{\vartheta}$ be an integral operator applied to a function u ; it is possible to have an operator equation like:

$$\hat{\vartheta}u = \hat{\xi}u + s$$

if $\hat{\xi}$ is an integral operator, we can take an approximation $u^{(0)}$ of u , apply $\hat{\xi}$ to it, and then:

$$\hat{\vartheta}u^{(1)} = \hat{\xi}u^{(0)} + s$$

so, we use $u^{(0)}$ as source to find $u^{(1)}$, which is a higher order approximation of the function; if $\hat{\xi}$ is an integral operator, the best the integral is estimated, the best the accuracy of our result will be.

Let us go on with the calculations on the differential equation; let us integrate $y_1''(x)$:

$$y_1''(x) = f(x)(x + 1)$$

so:

$$y_1'(x') = \int_0^{x'} f(x'')(x'' + 1) dx''$$

so, integrating this again:

$$y_1(x) = \int_0^x \int_0^{x'} f(x'')(x'' + 1) dx'' dx'$$

Let us remark that these integrals, evaluated in 0, return 0, since $y_1'(0) = 0$ and $y_1(0) = 0$ for hypothesis. We have two integrations; then, every time

that we have to find another term, we have to do other two integrations; so, for each higher order approximation, it is necessary to integrate twice a function; indeed, for instance, it is known that:

$$y_2''(x) = f(x)y_1(x)$$

so $f(x)y_1(x)$ has to be integrated twice, and this may be harder than what was suggested before for $y_0(x)$, which had an easy expression; this depends on $f(x)$ too, of course.

There are several methods aimed at solving in an approximated way a differential equation; an example is the Frobenius method, which consists of finding y as a Taylor series; the problem of this approach is that the Taylor series is a very good **local** approximation: it works only in the neighborhood of the point where the expansion is defined; instead, the perturbative series approach provides us with the **full picture**: maybe the approximation is not very good, but it is equally good in every point of the domain; so, we sacrifice the local information to obtain a decent approximation in the whole domain.

$$y(x) = \sum_{n=0}^{\infty} b_n x^n$$

this is very good in $x = 0$, but not outside of it, unless a huge number of terms is calculated.

2.1.3 Setting of a perturbation diffusion problem

Diffusion problems may be studied by means of a perturbation approach. An example of these problems is:

$$y''(x) = -e^{-x}y(x)$$

this equation is written in the form above, and we have a certain $f(x)$. This $f(x)$ is a sort of **diffusion coefficient**. Further details can be found in the book of Crank, **Mathematics on Diffusion**; this equation, in details, arises in the study of the conduction of heat in solids.

A perturbation method can be set up using the following idea:

$$t = 2\sqrt{\varepsilon}e^{-\frac{x}{2}}$$

since:

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$$

we have:

$$y''(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n''(x)$$

where:

$$y_0 = x + 1$$

for the usual boundary conditions. So:

$$\sum_{n=1}^{\infty} \varepsilon^n y_n''(x) = -\varepsilon e^{-x} \left[\sum_{n=1}^{\infty} \varepsilon^n y_n(x) + (x - 1) \right]$$

so, in general, it is possible to find that:

$$y_n''(x) = -e^{-x} y_{n-1}(x)$$

so, some kind of *iterative* method can be applied. The nonlinear term can be treated for instance expanding it as a Taylor series.

2.2 Regular and singular perturbation methods - multiple scale problems

The methods based on the use of a perturbation series can be divided in two classes:

- regular perturbation methods: it gives the exact solution when $\varepsilon \rightarrow 0$, but, for $\varepsilon \neq 0$, the series representation converges, in a non-trivial (different from a single point) interval, for some values of ε ; if this convergence exists, the perturbation series is said to be **regular**
- if we have the correct solution to the problem addressed with the perturbation series method only for $\varepsilon = 0$, then the perturbation method is called **singular**.

This means that we have two classes of methods but, for the second class, it is possible to solve the method only in the $\varepsilon = 0$ case; however, many questions can arise: is it useful to solve singular problems with perturbation methods? Is there a general way to identify singular problems, just looking at them? And, is it possible to manipulate a singular problem, to get somehow a non-singular one?

The solutions to these questions are: **multiscale methods**: by applying some scale changes, it is possible to tackle these problems.

Multiscale methods are based on the following idea: we can get a series representation non, in terms of powers of the parameters, but something else; indeed, if the problem can be solved in terms of a standard power series of ε , it means that the problem is regular; if it is not possible, the problem is singular.

In order to understand all of this, we are going to apply our idea to a specific problem:

$$\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0$$

Using ε as a parameter of this equation, and x as variable, this is an equation which admits 6 solutions; therefore, we want to find them.

Let us try to force $\varepsilon = 0$; in this case, the equation becomes:

$$-x^3 + 8 = 0$$

this equation has 3 roots only, rather than 6; these solutions are:

$$x_1 = 2 \quad x_2 = 2e^{j\frac{2\pi}{3}} \quad x_3 = 2e^{j\frac{4\pi}{3}}$$

These are three unperturbed roots; we can apply perturbation analysis to each of them and then find more accurate results. However, this provide us only with three out of six solutions; we missed three solutions!

What about the other three roots? For a part of the problem, standard techniques can not be applied, because we can not obtain the full solution of the problem only with the standard perturbation analysis. This is an indication of the fact that our problem is singular: since we can not get the full picture of the problem, using perturbation analysis, because the nature of the problem changed, well, this problem is probably singular: the application of $\varepsilon = 0$ caused some sort of degeneration of the problem from a sixth-order to a third-order problem.

What happened with the remaining three roots? For $\varepsilon \rightarrow 0$, what happens is that three roots are very close to the three numbers that we showed: x_1, x_2, x_3 ; the remaining three roots are huge numbers! Very large numbers!

But, how large? Which is the order of magnitude of x such as, for ε very small, we can obtain a polynomial equal to zero? If we can estimate, for each of these roots, the order of magnitude of x , we can apply some technique to transform this problem to another problem, being able to study all six solutions: this is the **scale change**.

In order to apply this approach, it is necessary to discuss the relative weight of the four terms of the equation: we have to study the orders of magnitude of the various terms; then, the idea may be to take two terms every time, and assume that they are comparable; if this equation is fulfilled by this requirement, we are fine.

The first assumption which we will try is:

$$\varepsilon^2 x^6 \sim \varepsilon x^4$$

this means that:

$$x = \frac{1}{\sqrt{\varepsilon}}$$

so, we are saying that:

$$x = O(\varepsilon^{-\frac{1}{2}})$$

with this condition, we can say that, substituting the behavior of $x^6 \propto \varepsilon^{-6/2}$:

$$\varepsilon^2 x^6 \propto \varepsilon^2 \varepsilon^{-3} = \varepsilon^{-1}$$

so:

$$\varepsilon^2 x^6 = O(\varepsilon^{-1})$$

this term goes as $O(\varepsilon^{-1})$; let us see what happens for other terms:

$$\varepsilon x^4 = O(\varepsilon \varepsilon^{-2}) = O(\varepsilon^{-1})$$

If x is of the order of magnitude of $\varepsilon^{-1/2}$, we have that $\varepsilon^2 x^6$ and εx^4 are both of order $O(\varepsilon^{-1})$. What about x^3 ?

$$x^3 = O(\varepsilon^{-\frac{3}{2}})$$

so, x^3 dominates with respect to ε^{-1} , for small values of ε ; therefore, this term can not equal zero, because no other terms can “compensate” it. This means that the first assumption is wrong, and we have to try something else. Indeed, by adding terms of order ε^{-1} , for $\varepsilon \rightarrow 0$, x^3 becomes too large, and nothing can compensate it.

Another assumption which can be done is:

$$\varepsilon x^4 \sim x^3$$

which becomes:

$$x = O(\varepsilon^{-1})$$

in this case:

$$x^3 = O(x^{-3})$$

but:

$$\varepsilon^2 x^6 = O(\varepsilon^{-6} \varepsilon^2) = O(\varepsilon^{-4})$$

and:

$$\varepsilon x^4 = O(\varepsilon \varepsilon^{-4}) = O(\varepsilon^{-3})$$

again, there is a dominant term, alone, which can not be compensated by any other term.

The third attempt is:

$$\varepsilon^2 x^6 \sim 8 \implies x = O(\varepsilon^{-\frac{1}{3}})$$

indeed, $8 = O(1)$. So:

$$\varepsilon x^4 = O(\varepsilon^{-\frac{4}{3}+1}) = O(\varepsilon^{-\frac{1}{3}})$$

So, this term is dominating, but what about x^3 ? Well:

$$x^3 = O(\varepsilon^{-1})$$

so this is dominating on the previous ones, and it can not be compensated, since it is alone.

It makes no sense to say that $x^3 \sim 8$; indeed, in this case, ε plays no role in our equation.

Another guess which we can try is:

$$\varepsilon^2 x^6 \sim x^3 \implies x = O(\varepsilon^{-\frac{2}{3}})$$

in this case:

$$x^3 = O(\varepsilon^{-2})$$

then:

$$8 = O(1)$$

and:

$$\varepsilon x^4 = O(\varepsilon^{1-\frac{8}{3}}) = O(\varepsilon^{-\frac{5}{3}})$$

The number $\frac{5}{3}$ is a little bit less than 2, so, now, the dominating term is ε^{-2} . This means that the terms that we left out are irrelevant with respect to the considered ones. So, since now the dominant terms are two, and the term with lower asymptotic order is alone, we have found a solution: we can use this to move on! Because, now the dominant term can compensate each other !!!

The following step is the application of a variable change:

$$y\varepsilon^{-2/3} = x$$

so, the equation becomes:

$$\varepsilon^{-2}y^6 - \varepsilon^{-5/3}y^4 - \varepsilon^{-2}y^3 + 8 = 0$$

which becomes:

$$y^6 - \varepsilon^{\frac{1}{3}}y^4 - y^3 + 8\varepsilon^2 = 0$$

Now, $\varepsilon = 0$ is not destroying the six roots: we have a **regular problem**.

To sum up, what we done was to “measure00 the order of magnitude, and then “change the scale”. Indeed, for $\varepsilon = 0$, we have:

$$y^6 - y^3 = 0$$

so:

$$y^3(y^3 - 1) = 0$$

so:

$$y = 0$$

is a solution three times: these are the “small” solutions which we have found without applying the multiscale method. Then:

$$y = 1, y = e^{j\frac{2}{3}\pi}, y = e^{j\frac{4}{3}\pi}$$

are the remaining three solutions which we were trying to find; to these three solutions we can apply our perturbation series, or something similar; indeed:

$$x(\varepsilon) = \sum_{n=0}^{\infty} a_n (\varepsilon^{\frac{1}{3}})^n$$

this is no longer a power series; this, because this is a singular perturbation problem; in this case, it is not possible to represent the solutions as power series. Indeed, the smallest power of ε that we have in our problem is $\varepsilon^{1/3}$, and so the series has to start with it.

2.2.1 Example: differential equation

Let us consider a differential equation:

$$\begin{cases} \varepsilon y'' - y' = 0 \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

If $\varepsilon \rightarrow 0$, the differential problem degenerates in an algebraic problem; we can suppose that it is not possible to write $y(x)$ as a perturbation series:

$$y(x) \neq \sum_{n=0}^{\infty} y_n(x) \varepsilon^n$$

So, which is the exact solution? Which perturbation analysis can provide us with a correct solution? Trying to put $\varepsilon = 0$, the equation becomes:

$$y'(x) = 0$$

which is a constant; but, this constant should equal zero for $x = 0$ and 1 for $x = 1$; this has no sense. This means that this solution can not even satisfy the two boundary conditions.

Using the exact analysis with the characteristic polynomial, we have:

$$\varepsilon\lambda^2 - \lambda = 0$$

so:

$$(\varepsilon\lambda - 1)\lambda = 0$$

which is:

$$\lambda = 0$$

$$\lambda = \frac{1}{\varepsilon}$$

This gives two exponentials:

$$y(x) = A + be^{\frac{x}{\varepsilon}}$$

Which is:

$$y(0) = 0 \implies A + B = 0 \implies A = -B$$

so:

$$y(x) = -B + Be^{\frac{x}{\varepsilon}} = 1$$

so:

$$B(e^{\frac{1}{\varepsilon}} - 1) \implies B = \frac{1}{e^{\frac{1}{\varepsilon}} - 1}$$

so, finally:

$$y(x) = \frac{1 - e^{\frac{x}{\varepsilon}}}{1 - e^{\frac{1}{\varepsilon}}}$$

If we draw this curve for decreasing values of ε , it tends to a constant almost everywhere, and it jumps to 1 in a very small interval. This behavior is

called “boundary layer”: everything happens in the region where the function varies.

In order to use the two-scales method, we need **asymptotic matching**: from the boundary layer on, we need one scale; then, for the other one, another; then, the two solutions have to be matched.

2.2.2 Perturbations on differential equations defined on an infinite interval

Let us consider, now, the following problem:

$$\begin{cases} y'' + (1 - \varepsilon x)y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

If this problem is given in a $[0, L]$ interval, this is a regular problem, and so in order to solve it it is possible to apply the standard perturbation theory. However, if the interval in which this problem has to be solved is $[0, \infty)$, it is not possible to use standard methods.

In perturbation analysis, ε can be assumed to be a small number; therefore, if we realize that, we can suppose that $\varepsilon \rightarrow 0$, and so the equation becomes:

$$y_0'' = -y_0$$

which is easy to be solved: this is the zero order approximation of the solution. But, to obtain further terms, we should have something like:

$$y(x) \simeq y_0(x) + \varepsilon y_1(x)$$

we obtained y_0 , but how can we obtain y_1 ?

An idea may be to take the previous expression of $y(x)$ and to differentiate it twice, then to substitute it in the differential equation:

$$y'(x) = y_0'(x) + \varepsilon y_1'(x)$$

and:

$$y''(x) = y_0''(x) + \varepsilon y_1''(x)$$

so, by substituting:

$$y_0'' + \varepsilon y_1'' + y_0 + \varepsilon y_1 - \varepsilon x(y_0 + \varepsilon y_1) = 0$$

by setting $y_0'' + y_0 = 0$, thing which is valid because y_0'' is a solution of that equation, the problem becomes corresponding to the first order coefficient in ε , equal to zero: it is simply necessary to equate the coefficient of ε to zero; so:

$$y_1'' + y_1 - x y_0 = 0$$

There is another philosophy to solve problems of this kind: let us assume to have a differential equation like:

$$y'' + (1 - x)y = 0$$

experience shows us that x is damaging us: it is what makes this problem hard! Therefore, we can **force** the presence of ε as a factor of x , and then at the end of the calculations set $\varepsilon = 1$. This allow us to use what we just learned: the 0-order solution is again $y_0'' + y_0 = 0$. Since y_0 is supposed to be an approximation of the solution, it is possible to compute the “complicated part” of the equation using the approximation:

$$y'' + y - \varepsilon x y = 0$$

substituting:

$$y_1'' + y_1 - \varepsilon x y_0 = 0$$

This problem has a peculiarity: y_0 is known, therefore it is a source of the problem.

These are the two ways of reaching this point; probably the first one is easier.

Let us complete the solution: relatively to the differential equation in y_0 , the characteristic polynomial is:

$$\lambda^2 + 1 = 0$$

therefore:

$$y_0(x) = A \cos x + B \sin x$$

since $y(0) = 1$, we have:

$$A + 0 = 1 \implies A = 1$$

then, since:

$$y'_0(0) = 1 \implies -A \times 0 + B \times 1 = 0 \implies B = 0$$

so

$$y_0(x) = \cos x$$

Now, let us write the perturbation series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \varepsilon^n$$

so:

$$y''(x) = \sum_{n=0}^{\infty} y''_n(x) \varepsilon^n$$

so:

$$\sum_{n=0}^{\infty} y''_n(x) \varepsilon^n + \sum_{n=0}^{\infty} y_n(x) \varepsilon - x \sum_{n=0}^{\infty} y_n(x) \varepsilon^{n+1} = 0$$

by forcing coefficients of the same order of ε to equal zero, we obtain the following equation:

$$y''_n(x) + y_n(x) - x y_{n-1}(x) = 0$$

since we solved the case $n = 0$, it is possible to substitute it and to obtain the $n = 1$ equation:

$$y''_1(x) + y_1(x) - x \cos x = 0$$

This differential equation has a homogeneous part of the solution, which is related to the own response of the system, to the kernel of the differential operator, and then, added to that, the response of the source. We can find that:

$$y_1(x) = \frac{1}{4}x^2 \sin x + \frac{1}{4}x \cos x - \frac{1}{4} \sin x$$

and then, we can apply the same for y_2 :

$$y_2'' + y_2 - xy_2 = 0$$

and it is possible to prove that:

$$y_2(x) = -\frac{1}{32}x^4 \cos x + \frac{5}{48}x^3 \sin x + \frac{7}{16}x^2 \cos x - \frac{7}{16}x \sin x$$

these proofs are done in the homeworks.

The solution which we showed works only for small values of x : indeed, even if ε is very small, if x becomes too large, it is not possible to say that $x\varepsilon$ is still small! For large values of x , there is no way to have a convergent perturbation series!

So, how can we tackle this problem using perturbation methods? Well, the solution, is to use a multi-scale approach; indeed, if $x \sim \frac{1}{\varepsilon}$, this problem becomes singular, and so it is necessary to develop, in the large scale, another perturbation method, which is able to work also for the unbounded domain.

2.2.3 Application example: Schrödinger equation

Let us consider the 1d Schrödinger equation, in planar geometry:

$$\left[-\frac{d}{dx^2} + u(x) - E \right] y(x) = 0$$

the derivative is said to be “diffusion operator”; $y(x)$ is connected to the probability to find a particle between the points x and $x + dx$.

The potential u is what bounds the possibility of motion of the particle; therefore, if $u(x)$ is very big at infinity, the particle can not escape; therefore, we want:

$$\lim_{|x| \rightarrow \infty} u(x) \rightarrow +\infty$$

Therefore:

$$\lim_{|x| \rightarrow \infty} y(x) = 0$$

the particle can not escape towards infinite.

Another condition which may be required is the regularity of the solution, which means a certain number of finite derivatives.

A remark: this is an homogeneous problem, because no forcing term is present; the objective of this problem is to determine E , which is the set of possible energy levels of the particle. This is a Sturm-Liouville problem, so the theory behind it is very known. However, if u has complicated expressions, the problem can not be solved analytically. E is the eigenvalue of the problem.

Let us try to apply a perturbation approach: since the complications of the problem come from $u(x)$, the obvious suggestion is to put ε in front of it; however, if $\varepsilon \rightarrow 0$ in this case, the problem becomes the problem of a particle which is no longer moving in an energy field. So, probably, this approach is too hardcore.

But, let us assume that $u(x)$ can be divided in two parts:

$$u(x) = v(x) + w(x)$$

where $v(x)$ is an “easy” term, and $w(x)$ a “hard” term, which complicates the expressions; with only $v(x)$, we assume to have a problem which can be solved analytically. So, it is possible to consider:

$$v(x) + w(x) \longrightarrow v(x) + \varepsilon w(x)$$

and so, the problem becomes:

$$\left[-\frac{d}{dx^2} + v(x) + \varepsilon w(x) - E \right] y(x) = 0$$

Now, let us consider $\varepsilon = 0$, and let us choose, for instance:

$$v(x) = \frac{x^2}{4}$$

this is just an example of problem, just to have something which can be discussed. The zero order equation is:

$$-\frac{d^2y}{dx^2} + \frac{x^2}{4}y - Ey = 0$$

This is an eigenvalue equation; in Abramowitz it is possible to find the solution for the equation: the eigenvalue E assumes only a discrete number of values identified by the index m :

$$E_m = m + \frac{1}{2}$$

where m is an integer number. The fact that this is finite is the starting point of quantum mechanics, for which the energy levels which a particle can assume are quantized.

Again from Abramowitz it is possible to see that, for this specific expression of the easy part of the potential $v(x)$:

$$y_m(x) = e^{-\frac{x^2}{4}} H_m(x)$$

where $y_m(x)$ are the eigenfunctions of the problem; H_m are the Hermite polynomials. For instance, $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$.

This is only a zero order solution, so all these eigenvalues and eigenfunctions are an approximation of the actual problem, because we have only kept into account $v(x)$, but not $w(x)$!

Therefore, in order to improve the approximation of each m -th eigenvalue and eigenfunction, let us write, for a specific eigenvalue E , its perturbation series:

$$E = \sum_{n=0}^{\infty} E_n \varepsilon^n$$

the solution of the problem, i.e. the eigenfunction, is:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \varepsilon^n$$

(this is the eigenfunction associated to the eigenvalue E). So, substituting in the original equation:

$$-\sum_{n=0}^{\infty} y_n''(x) \varepsilon^n + v(x) \sum_{n=0}^{\infty} y_n(x) \varepsilon^n - w(x) \sum_{n=0}^{\infty} y_n(x) \varepsilon^{n+1} - \sum_{n=0}^{\infty} E_n \varepsilon^n \sum_{n=0}^{\infty} y_n(x) \varepsilon^n = 0$$

As usual, let us satisfy this equation by equating to 0 the coefficient of the n -th power of ε :

$$-y_n''(x)v(x)y_n(x) + w(x)y_{n-1}(x) - \sum_{j=1}^n E_j y_{n-j}(x) - E_0 y_n = 0$$

where the last term comes from the Cauchy theorem for the product of two series.

Owing to the presence of the potential, we require that y_n , for big values of x , approach 0; keeping into account this condition too, this is the set of equations that we have to solve; the unknown of the problem are the coefficients of the perturbation series, y_n ; the highest order in the series is $n - 1$, since the summations are calculated starting from $j = 1$; finally, we derived a connection between all the terms prior to the n -th one, from y_0 to y_{n-1} .

The objective of all these calculations is to find the eigenvalues and eigenvectors for the n -th order; indeed, up to the $n - 1$ th, everything is known. So:

$$-y_n'' + v(x)y_n(x) - E_0 y_n(x) = -w(x)y_{n-1}(x) + \sum_{j=1}^n E_j y_{n-j}(x)$$

This is a non-homogeneous problem, because it has a source term; this problem can be solved by means of the **method of order reduction**: defining $y_n(x)$ as a function of $y_0(x)$ times a certain new unknown $F_n(x)$, we can write:

$$f_n(x) = y_0(x)F_n(x)$$

This approach is absolutely general, and it can also be applied to other situations or equations. Now, we can find its derivatives, using Leibnitz's rule:

$$y_n'(x) = y_0'(x)F_n(x) + y_0(x)F_n'(x)$$

and:

$$\begin{aligned} y_n''(x) &= y_0''(x)F_n(x) + y_0'(x)F_n'(x) + y_0'(x)F_n'(x) + y_0(x)F_n''(x) = \\ &= y_0''(x)F_n(x) + 2y_0'(x)F_n'(x) + y_0(x)F_n''(x) \end{aligned}$$

The calculations of these derivatives is useful, to substitute these expressions in the previous equation:

$$-y_0''F_n - 2y_0'F_n' - y_0F_n'' + v(x)y_0F_n - E_0y_0F_n = -wy_0F_{n-1} + \sum_{j=1}^n E_jy_0F_{n-j}$$

This, simplifies our problem; indeed, now we have, collecting all the terms in F_n :

$$F_n(-y_0'' + v(x)y_0 - E_0y_0)$$

but, since y_0 is the solution of the equation, what is inside the parentheses equals zero. For what concern the remaining terms, we have, changing signs at both terms:

$$2y_0'F_n' + y_0F_n'' = wy_0F_{n-1} - \sum_{j=1}^n E_jy_0F_{n-j}$$

let us multiply both members times y_0 :

$$y_0(2y_0'F_n' + y_0F_n'') = y_0\left(wy_0F_{n-1} - \sum_{j=1}^n E_jy_0F_{n-j}\right)$$

but, it is possible to see that:

$$y_0(2y_0'F_n' + y_0F_n'') = \frac{d}{dx}(y_0^2F_n')$$

so, we obtained:

$$\frac{d}{dx}(y_0^2F_n') = w(x)y_0^2(x)F_{n-1}(x) - \sum_{j=1}^n E_jy_0^2(x)F_{n-j}(x)$$

Observing that:

$$y_n = F_ny_0 \implies F_n = \frac{y_n}{y_0}$$

so, it is possible to write:

$$y_0^2(x) \frac{d}{dx} \left(\frac{y_n}{y_0} \right) = y_0^2 \left(\frac{y_n'}{y_0} - \frac{y_0'}{y_0^2} y_n \right) = y_0y_n' - y_0'y_n$$

but, since when $|x| \rightarrow \infty$ goes to zero, and y'_n, y'_0 are finite (we ask for regularity in the solution!),

$$\lim_{|x| \rightarrow \infty} y_0^2 F'_n = 0$$

Now, let us integrate the equation from $-\infty$ to $+\infty$; this improper integral equals zero, because it equals the difference between the two limits of the integrand function in $\pm\infty$, which are both zero, as we have just proved. So:

$$\int_{-\infty}^{+\infty} \frac{d}{dx} (y_0^2 F'_n) dx = (y_0^2 F'_n) \Big|_{-\infty}^{+\infty} = 0$$

So, since integration should be applied also the right term, we have just shown that:

$$\int_{-\infty}^{+\infty} y_0^2(x) \left[w(x) F_{n-1}(x) - \sum_{j=1}^n E_j F_{n-j}(x) \right] dx = 0$$

From this integral, let us now separate the terms of order n from the remaining ones; the only part where the n -th term appears is the sum, so, this becomes:

$$\int_{-\infty}^{+\infty} y_0^2(x) \left[w(x) F_{n-1}(x) - \sum_{j=1}^{n-1} E_j F_{n-j}(x) \right] dx + \int_{-\infty}^{\infty} y_0 E_n F_0(x) dx = 0$$

but $F_0(x) = \frac{y_0(x)}{y_0(x)} = 1$; so:

$$\int_{-\infty}^{\infty} y_0 E_n F_0(x) dx = E_n \int_{-\infty}^{+\infty} y_0^2(x) dx$$

So, we just proved that:

$$E_n \int_{-\infty}^{+\infty} y_0^2(x) dx = \int_{-\infty}^{+\infty} y_0^2(x) \left[w(x) F_{n-1}(x) - \sum_{j=1}^{n-1} E_j F_{n-j}(x) \right] dx$$

which means that:

$$E_n = \frac{\int_{-\infty}^{+\infty} y_0^2(x) \left[w(x)F_{n-1}(x) - \sum_{j=1}^{n-1} E_j F_{n-j}(x) \right] dx}{\int_{-\infty}^{+\infty} y_0^2(x) dx}$$

Now, some additional simplification can be introduced; indeed:

$$F_{n-1} = \frac{y_{n-1}}{y_0}$$

so, simplifying y_0 , we obtain:

$$E_n = \frac{\int_{-\infty}^{+\infty} y_0(x) \left[w(x)y_{n-1}(x) - \sum_{j=1}^{n-1} E_j y_{n-j}(x) \right] dx}{\int_{-\infty}^{+\infty} y_0^2(x) dx}$$

This formula is very interesting; indeed, in it it is possible to read that to compute the n -th order approximation of the eigenvalue E , E_n , it is just necessary to know the previous coefficients of the approximation (up to the $n - 1$ term), and of the eigenfunctions. This integral can not be calculated analytically, but in most cases it can be solved by means of a quadrature formula.

Perturbation of the eigenfunctions

We are going to try to find a similar formulation for what concerns the elements of the perturbation series of the eigenfunctions. Going back, we proved:

$$\frac{d}{dx} \left[y_0^2(x) F_n'(x) \right] = y_0^2(x) \left[w(x) F_{n-1}(x) - \sum_{j=1}^n E_j F_{n-j}(x) \right]$$

instead of integrating in the entire domain, let us integrate from $-\infty$ to a certain parameter t ; the result is:

$$y_0^2(t) F_n'(t) - y_0^2(-\infty) F_n'(-\infty) = \int_{-\infty}^t y_0^2(s) \left[w(s) F_{n-1}(s) - \sum_{j=1}^n E_j F_{n-j}(s) \right] ds$$

but $y_0^2(-\infty) = 0$ (this is a limit!). So, it is possible to isolate $F_n(x)$ and to divide both terms times $y_0^2(x)$, then to integrate, obtaining:

$$F_n(x) \int_a^x \frac{1}{y_0^2(t)} \int_{-\infty}^t y_0(s) \left[w(s)y_{n-1}(s) - \sum_{j=1}^n y_{n-j}(s) \right] ds dt$$

(this, by doing the same step which has been done at the end of the previous subsection). In this equation, everything is completely defined, but the evaluation of this expression can be very complicated.

Chapter 3

Perturbation theory of integral quantities

One important point, when applying mathematical methods to physical problems, is to identify which are the **important** parameters: many times, having too information can be a problem, because information has to be handled, processed, in order to derive what is really needed to solve the physical problem. Integral quantities are very important in physics, because they provide the researcher with a summary of the information: by integrating we lose local information, but we have less quantities to study: it is like considering an average, instead of a function.

3.1 Mathematical fundamentals

In this section we will set the mathematical notation of this chapter¹. We will identify with $\hat{\vartheta}$ our operators; as operators, we usually mean **integral operators**, as the name of the chapter is suggesting. $\hat{\vartheta}$ has a domain, which is a set of functions which can be applied to it; when this is possible, we identify the application of $\hat{\vartheta}$ to a certain set of functions f belonging to its domain with the notation

$$\hat{\vartheta}f$$

¹A reference for operator theory oriented to perturbation theory is: Kato T. - Perturbation Theory for Linear Operators

Now, let us consider $a \in f$, where a is a single function; let b be another function, also of another space; it is possible to define (b, a) as the **inner product** of the two functions. In order to get some confidence with the concept of inner product, let us consider an example related to \mathbb{R}^2 ; to this aim, let us consider the following examples:

$$\begin{aligned}\underline{u} &= u_1\hat{e}_1 + u_2\hat{e}_2 && \text{direct vector} \\ \underline{v} &= v_1\hat{e}_1 + v_2\hat{e}_2 && \text{adjoint vector}\end{aligned}$$

where \hat{e}_1 and \hat{e}_2 are the unit vectors in which the two vectors are represented, and (u_1, u_2) , (v_1, v_2) are the components of the vector of the direct space and of the adjoint space respectively. An example of inner product between these two vectors, $(\underline{v}, \underline{u})$ can be defined as:

$$(\underline{v}, \underline{u}) \triangleq u_1v_1 + u_2v_2$$

In other words, the inner product, starting from an element (a function) of the direct space, and from one of the adjoint space, produces a number.

Now, let us assume that the application of the operator \hat{v} to a function belonging to the functional space f maps the function in the space f itself. Then, let us consider another space, g ; so, it may be interesting to find:

$$(g, \hat{v}f)$$

Conversely, it is also possible, give \hat{v}^\dagger , operator which can be applied to functions belonging to the space g , to define:

$$(\hat{v}^\dagger g, f)$$

These two situations, right now, are separated. But, if:

$$(\hat{v}^\dagger g, f) = (g, \hat{v}f)$$

so, \hat{v}^\dagger is said to be the **adjoint operator** of \hat{v} .

In other words, given a function belonging to the space g , and one to the space f , it is possible to move the operator (doing the adjoint) without changing the result.

3.2 Eigenvalue problems

Given \hat{v} any linear, differential/integral operator, x an element of the phase space (it may be a vector):

$$\hat{v}y(x) - \lambda y(x) = 0$$

in this case, λ is the eigenvalue of the problem. In these notes we will consider $\lambda \in \mathbb{R}$; we will focus on one particular value of λ , and, since it is an eigenvalue for assumption, it will belong to the discrete part of the spectrum. Finally, let us consider:

$$\hat{v} = \hat{v}_0 + \delta\hat{v}$$

the operator \hat{v} consists of two parts: \hat{v}_0 is an unperturbed operator, while $\delta\hat{v}$ is the perturbation from \hat{v}_0 . Since this is a perturbation, we may apply perturbation analysis to this problem, and assume that:

$$\delta\hat{v} = \varepsilon\hat{\Phi}$$

where $\hat{\Phi}$ is another operator. In order to obtain the first-order approximation of the perturbed eigenvalue, it is necessary to solve the following problem:

$$(\hat{v}_0 + \varepsilon\hat{\Phi})y(x) - \lambda y(x) = 0$$

If $\varepsilon = 0$ we have the unperturbed problem, so the one involving only the unperturbed operator \hat{v}_0 :

$$\hat{v}_0 y_0(x) - \lambda_0 y_0(x) = 0$$

let us assume that this problem can be solved, and let us assume to know the solution $y_0(x)$ and λ_0 . Now, we want to obtain a more accurate approximation of the eigenvalue, so λ_1 ; to this aim, it is possible to apply a perturbation approach, and to write the perturbation series, truncated to the first power in ε , for both the eigenvalue and the eigenfunction:

$$y(x) \simeq y_0(x) + \varepsilon y_1(x) \qquad \lambda(x) \simeq \lambda_0 + \varepsilon \lambda_1$$

therefore, by substituting this in the equation:

$$(\hat{\vartheta}_0 + \varepsilon \hat{\Phi})(y_0 + \varepsilon y_1) - (\lambda_0 + \varepsilon \lambda_1)(y_0 + \varepsilon y_1) = 0$$

now, we expand this equation:

$$\hat{\vartheta}_0 y_0 + \varepsilon \hat{\vartheta}_0 y_1 + \varepsilon \hat{\Phi} y_0 + \varepsilon^2 \hat{\Phi} y_1 - \lambda_0 y_0 - \lambda_0 \varepsilon y_1 - \varepsilon \lambda_1 y_0 - \varepsilon^2 \lambda_1 y_1 = 0$$

now, grouping as usual all the 0-order and 1-order terms, it is possible to define two equations; for zero order terms:

$$\hat{\vartheta}_0 y_0 - \lambda_0 y_0 = 0$$

this is nothing new: this is the zero order problem, and this is assumed to be known. Therefore, it can be canceled from the equation. Then, the ε^2 terms are neglected, because they are not important in the first order approximation; finally, what remains is the following equation:

$$\varepsilon \left[\hat{\vartheta}_0 y_1 + \hat{\Phi} y_0 - \lambda_0 y_1 - \lambda_1 y_0 \right] = 0$$

so, what is inside the parentheses should be equated to zero. The interesting thing of this equation is the fact that $\hat{\Phi}$ is applied to y_0 , which is known; therefore, since the “bad” operator, the perturbation, is applied only to y_0 , well, this is simply a source of the operation equation.

There are cases in which we are not interested in knowing y_1 , so in improving the approximation for the eigenfunction, but only in improving the accuracy with which we know the eigenvalue. The interesting thing is the fact that it is possible to obtain λ_1 without evaluating y_1 . The idea is: first, we have to multiply this equation times something; then, y_1 terms become in some sense orthogonal to the multiplication terms, and so we eliminate its presence from the equation. Now the question is: on which function is it possible to perform a projection of this equation in order to get rid of the y_1 contribution? Well, the answer is quite easy: if we have solved the unperturbed adjoint problem:

$$\hat{\vartheta}_0^\dagger y_0^\dagger(x) - \lambda_0^\dagger y_0^\dagger(x) = 0$$

where it can be shown by means of a general theorem that:

$$\lambda_0^\dagger = \lambda_0^*$$

so, the eigenvalue of the adjoint problem equals the complex conjugate of the eigenvalue of the direct problem. Assuming that also the adjoint problem is solved, the projection is performed by means of an inner product, which is realized by means of an integral. Since we are going to consider only integrals from now on, we are changing the local equation in an integral equation; therefore, we are forcing to equal zero not the local equation, in strong form, but only a weighted, averaged quantity; the importance of our quantities is weighted with y_0^\dagger :

$$\left(y_0^\dagger, \hat{\vartheta}_0 y_1 + \hat{\Phi} y_0 - \lambda_0 y_1 - \lambda_1 y_0 \right) = 0$$

exploiting the linearity of the inner product it is possible to obtain:

$$(y_0^\dagger, \hat{\vartheta}_0 y_1) + (y_0^\dagger, \hat{\Phi} y_0) - \lambda_0 (y_0^\dagger, y_1) - \lambda_1 (y_0^\dagger, y_0) = 0$$

let us focus on the following terms:

$$(y_0^\dagger, \hat{\vartheta}_0 y_1) - \lambda_0 (y_0^\dagger, y_1)$$

it is possible to move the operator from the first term, adjoining it:

$$(y_0^\dagger, \hat{\vartheta}_0 y_1) = (\hat{\vartheta}_0^\dagger y_0^\dagger, y_1)$$

so, we obtain:

$$(\hat{\vartheta}_0^\dagger y_0^\dagger, y_1) - \lambda_0 (y_0^\dagger, y_1)$$

using linearity:

$$(\hat{\vartheta}_0^\dagger y_0^\dagger - \lambda_0 y_0^\dagger, y_1)$$

but:

$$\hat{\vartheta}_0^\dagger y_0^\dagger - \lambda_0 y_0^\dagger = 0$$

because this is the adjoint eigenvalue problem relative to the zero order approximation! So, everything is 0, and the equation reduces to:

$$(y_0^\dagger, \hat{\Phi} y_0) - \lambda_1 (y_0^\dagger, y_0) = 0$$

which can be written as:

$$\lambda_1 = \frac{(y_0^\dagger, \hat{\Phi}y_0)}{(y_0^\dagger, y_0)}$$

This means that it is possible to solve only the unperturbed problem and the adjoint unperturbed problem, in order to refine the approximation of the eigenvalue of the problem perturbed by $\hat{\Phi}$. This, is done just by means of easy integrations.

Many times it is enough to have a first order approximation; however, there are also situations in which it is not possible to stop at the first term. An example where this happens is **nuclear reactors**: when we introduce the control rods, which are neutron absorbers, we reduce the chain reaction. Control rods represent a perturbation of the operator, where the operator is the “chain reaction”. If we go only to the first order, we compute the capability to change the multiplication of neutrons keeping into account the presence of the control rod, starting from an unperturbed system; however, since control rods represent a perturbation to this system, we can not start from an unperturbed situation to model this! The point is: the first order approximation, as we just proved, is not taking into account the presence of the perturbation itself and, sometimes, this can not be done, because, as in the nuclear engineering case study, the number of neutrons can not be calculated correctly.

3.3 Perturbation of integral quantities

Now we are going to focus on the perturbation of integral quantities, but no longer on eigenvalues. Let us consider a problem like:

$$\hat{\vartheta}y(x) + S(x) = 0$$

This may be an integral or a differential equation, and $y(x)$ is the solution of this operator equation; however, it is possible to be interested not in y for each point x of the phase state, but on some functional related to it; therefore, the problem which we are addressing is the inversion of the operator $\hat{\vartheta}$, but in some “global” sense. For instance, we may be interested in an integral quantity, like:

$$I = (w, y)$$

where w is something that we want to observe; for instance, it may be the total number of particles, or their importance in the reaction; w is called the “detector” of our problem, because it selects, it weights, among the values of y .

Also in this case it is possible to have an operator $\hat{\vartheta}$ which equals some unperturbed operator, plus a perturbation, just like in the eigenvalues case studied in the previous section:

$$\hat{\vartheta} = \hat{\vartheta}_0 + \delta\hat{\vartheta} = \hat{\vartheta}_0 + \varepsilon\hat{\Phi}$$

just like in the previous section, we will focus only on the first order approximation of the perturbation. Since in this case the only thing which we have is y , we are going to perform our perturbation analysis on y only:

$$y(x) = y_0(x) + \varepsilon y_1(x)$$

The problem without perturbation is:

$$\hat{\vartheta}_0 y_0(x) + S(x) = 0$$

we assume that this problem can be somehow solved. Then, the first order approximation of the problem is, substituting the perturbation series inside the equation:

$$(\hat{\vartheta}_0 + \varepsilon\hat{\Phi})(y_0 + \varepsilon y_1) + S = 0$$

Now, our objective is to calculate δI , not δy ; therefore, let us keep into account this objective while proceeding through the following steps. Just like in the previous section, the first step is the expansion of the previous equation:

$$\hat{\vartheta}_0 y_0 + \varepsilon\hat{\vartheta}_0 y_1 + \varepsilon\hat{\Phi}y_0 + \varepsilon^2\hat{\Phi}y_1 + S = 0$$

it is possible once again to remove the terms in ε^2 , and the zero order problem; therefore, the equation becomes:

$$\varepsilon \left[\hat{\vartheta}_0 y_1 + \hat{\Phi}y_0 \right] = 0$$

so, to satisfy this problem $\forall \varepsilon$, the equation is:

$$\hat{\vartheta}_0 y_1 + \hat{\Phi}y_0$$

since y_0 is known (we assume to have solved the zero order problem), this can be seen once again as an operator equation with a source.

Before going on with the projection, it is still possible to define, for the zero order problem of this situation, another zero order adjoint problem as:

$$\hat{\vartheta}_0^\dagger y_0^\dagger + S^\dagger = 0$$

A remark: since we have a source, we are not discussing eigenvalue problems; indeed, there is a theorem which suggests that, in this case, the homogeneous adjoint problem may have no solution; therefore, we added an adjoint source. As adjoint source, it is possible to choose the detector w :

$$\hat{\vartheta}_0^\dagger y_0^\dagger + w = 0$$

indeed, by this way, we are obtaining the same “miracle” that we obtained before; if we choose the detector, we obtain:

$$(y_0^\dagger, \hat{\vartheta}_0 y_1) + (y_0^\dagger, \hat{\Phi} y_0) = 0$$

where the second term is known; therefore:

$$(y_0^\dagger, \hat{\vartheta}_0 y_1) = -(y_0^\dagger, \hat{\Phi} y_0)$$

but, since

$$\hat{\vartheta}_0^\dagger y_0^\dagger = -w$$

we can write:

$$(y_0^\dagger, \hat{\vartheta}_0 y_1) = (\hat{\vartheta}_0^\dagger y_0^\dagger, y_1) = (-w, y_1)$$

so, substituting:

$$(-w, y_1) + (y_0^\dagger, \hat{\Phi} y_0) = 0$$

therefore:

$$(w, y_1) = (y_0^\dagger, \hat{\Phi} y_0)$$

where the right-hand side member is known. Now, since:

$$I = (w, y_0 + \varepsilon y_1) = (w, y_0) + \varepsilon(w, y_1) = I_0 + \varepsilon \delta I_1$$

we have that:

$$\delta I_1 = (w, y_1) = (y_0^\dagger, \hat{\Phi} y_0)$$

and, just by solving the zero order problem and the adjoint problem, it is possible to obtain also the first order approximation of the functional.

A remark: this is true only if we keep always the same detector; if we change the detector, we change the functional, and so for different detectors, it is always necessary to solve a new adjoint problem.

As usual, it is possible to discuss this result. When is the first order analysis sufficient to have satisfactory results? Well, it is possible to remove high order terms if the perturbation operator $\hat{\Phi}$ has a small norm compared to the unperturbed operator:

$$\frac{\|\hat{\Phi}\|}{\|\hat{\vartheta}_0\|} \text{ should be small}$$

What is limiting the validity of our perturbation analysis is the norm of the operators.

3.4 Notes on sensitivity

An exercise which can be done at this point is, given any 2×2 matrix, for instance with real eigenvalues, to apply perturbation analysis to the perturbations of this matrix, and calculate how eigenvalues are perturbed by the perturbation of the operator; in this case:

$$\hat{\vartheta}_0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then, let $\hat{\Phi}$ be a perturbation matrix:

$$\hat{\Phi} = \begin{bmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{bmatrix}$$

for instance, given λ_1 the first eigenvalue of the operator $\hat{\vartheta}_0$, it is possible to estimate its sensitivity to a_{11} , by defining a perturbation operator acting only on a_{11} :

$$\hat{\Phi}_{11} = \begin{bmatrix} \delta a_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

if $\delta a_{11} = 1\%$, it is possible to evaluate $S_{a_{11}}^{\lambda_1}$, which is the sensitivity of λ_1 on a variation of a_{11} , as:

$$S_{a_{11}}^{\lambda_1} = \frac{\delta \lambda_1}{\delta a_{11}}$$

Sensitivity analysis is useful because, in general, a problem is defined by a certain number of parameters; in this case we have the operator $\hat{\vartheta}_0$ which has 4 parameters, which are the elements of the matrix representing the operator. Different parameters of the operator, of the model, produce different results. Through perturbation theory it is possible to estimate the perturbation on the result starting from perturbations of the parameters. In general, it is possible to defined sensitivity of the j -th output of our model r_j on the i -th parameter of the model p_i as:

$$S_{p_i}^{r_j} = \frac{\delta r_j}{\delta p_i}$$

But it is possible to be interested in relative values:

$$S_{ij} = \frac{\frac{\delta r_j}{r_j}}{\frac{\delta p_i}{p_i}}$$

in this case, the perturbations have been normalized to the nominal values; since we have the indices j and i , this constitutes a matrix, which quantifies the effect of the i -th parameter on the j -th result. For instance, in the case study, we have two outputs, which are the eigenvalues λ_1 and λ_2 , and four inputs, which are the matrix elements; therefore, the sensitivity matrix will be a matrix with two rows (one per each output) and four columns (one for each parameter):

$$\underline{S} = \begin{bmatrix} S_{p_1}^{r_1} & S_{p_2}^{r_1} & S_{p_3}^{r_1} & S_{p_4}^{r_1} \\ S_{p_1}^{r_2} & S_{p_2}^{r_2} & S_{p_3}^{r_2} & S_{p_4}^{r_2} \end{bmatrix} = \begin{bmatrix} S_{a_{11}}^{\lambda_1} & S_{a_{12}}^{\lambda_1} & S_{a_{21}}^{\lambda_1} & S_{a_{22}}^{\lambda_1} \\ S_{a_{11}}^{\lambda_2} & S_{a_{12}}^{\lambda_2} & S_{a_{21}}^{\lambda_2} & S_{a_{22}}^{\lambda_2} \end{bmatrix}$$

indeed, in our case, $r_1 = \lambda_1$, $r_2 = \lambda_2$, $p_1 = a_{11}$, $p_2 = a_{12}$, $p_3 = a_{21}$, $p_4 = a_{22}$. Let us remark that, in general, sensitivity matrices are **not** square matrices.

Chapter 4

Asymptotic matching

The objective of this chapter is to solve differential equations in non-homogeneous problems¹.

The first problem is the fact that the coefficients depend on the independent variable of the problem; therefore, the differential equation is something like:

$$f(y'', y', x) = 0$$

where the coefficients are depending on x too. An example of these differential equations is:

$$y'' + \sqrt{x}y' + 2x = 0$$

in this equation, \sqrt{x} is a coefficient, and $2x$ is the source of the problem. In this case the coefficient is varying with continuity, but there are cases in which the coefficient of the differential equation is piecewise constant, therefore it has an abrupt variation in one point, and then inside some intervals it is constant. In this situation, in each part of the domain where the coefficient is constant, the differential equation can be solved as standard ODE with constant coefficients.

The problem of this approach is the fact that we are solving two problems, solved in two subintervals $[a, c]$, and $[b, c]$, are separated: nothing is relating them, and this is wrong, since the differential equation is solved in the entire domain $[a, b]$. Therefore, our objective is to force continuity of the solution in

¹As a reference for the differential equations subject, the “Tricomi - Differential Equations“ (Dover) book is good.

the discontinuity point: we want to **match** the two solutions in the common point between the two adjacent intervals. This procedure is called **point matching**: we have to match the solutions in a single point.

4.1 Point matching example

An easy example of what has been stated in the previous equation is:

$$y''(x) - y(x) = e^{-|x|}$$

in this example, only the **source** has a break in $x = 0$. If we want to study this problem $\forall x \in \mathbb{R}$, it is necessary to separate it in two subproblems:

$$\begin{cases} x \geq 0 & y'' - y = e^{-x} \\ x \leq 0 & y'' - y = e^x \end{cases}$$

The first step, valid for both these equations, is the solution of the homogeneous equation:

$$y_h'' - y_h = 0$$

using the usual ODE theory:

$$y_h(x) = ae^x + be^{-x}$$

If we want to enforce regularity,

$$\lim_{|x| \rightarrow \infty} y_h(x) = 0$$

if we focus on $x \rightarrow -\infty$, we have $b = 0$; therefore:

$$y_h(x) = ae^x$$

The homogeneous solution is resonant to the source, therefore we multiply the source times x , to define the form of the particular solution, in order to eliminate the dependence (resonance) with the source:

$$y_p(x) = Kxe^x$$

So, let us calculate the derivatives of this expression:

$$y'_p(x) = Ke^x + Kxe^x = Ke^x(x + 1)$$

and:

$$y''_p(x) = Ke^x + Ke^x(x + 1) = Ke^x(x + 2)$$

Since:

$$y''_p(x) - y_p(x) = e^x$$

we have:

$$Ke^x(x + 2) - Ke^x = e^x \implies 2K = 1 \implies K = \frac{1}{2}$$

Therefore:

$$y_p(x) = \frac{1}{2}xe^x$$

and, finally:

$$y(x) = y_h(x) + y_p(x) = ae^x + \frac{1}{2}xe^x$$

It is possible to repeat very similar steps for the $x \geq 0$ case, obtaining:

$$y(x) = be^{-x} - \frac{1}{2}xe^{-x}$$

Up to this point, the two solutions are disjoint; but there are still the a and b parameters, which are degrees of freedom of our problem; in order to complete the problem, it is necessary to **patch** the two solutions. Since $y(x)$ comes from equations with a physical meaning, $y(x)$ has a physical meaning too: therefore the conditions at the common point may be continuity, differentiability or whatever is driven by the physics of the problem. Let us require, for this specific example, the continuity of both the function and its first derivative:

$$\begin{cases} y(0^-) = y(0^+) \\ y'(0^-) = y'(0^+) \end{cases}$$

If we apply these conditions, we obtain:

$$a = b = -\frac{1}{2}$$

The operation which we have just performed is a **matching**. The idea of matching is: since in part of the domain we can apply a certain method of solution of the differential equation, for instance a perturbation analysis, and in another part of the domain another method, for instance a multi-scale method, since the differential equation is defined on a hemi-infinite domain, it is possible to join, to match the solutions obtained with two different methods.

Up to this point, we were discussing **point matching**, *i.e.* matching the two solutions in a single point. Now, let us consider, for instance, a differential equation defined within an interval $[a, b]$; let us consider $a \leq b'$, $a' \leq b$; then, let us assume that, for this particular problem, we can apply a certain method in a region $[a, a']$, for instance perturbation analysis; instead, on $[b, b']$, maybe it is possible to provide an analytical solution to the differential equation. It is possible that $[b', a']$ is an interval (if $b' \neq a'$), so in this case it is necessary to match functions in an entire interval, instead that on a single point; this is a generalization of what we have discussed in point matching: indeed, point matching is a sort of degeneration of what happens on an interval.

Matching two solutions on an interval can be useful also under another point of view; indeed, considering for instance a transport problem, close to the source it is necessary to use a complex transport model to take into account every effect while, far from the source, it is possible to reduce the transport model to a diffusion model, which is easier to handle. This is the basic idea behind **hybrid methods**.

4.2 Reference solution for the asymptotic matching solution

In order to explain the idea of asymptotic matching, it will be applied directly to a differential problem; therefore, we are going to solve the following differential problem:

$$\begin{cases} y'(x) + \left(\varepsilon x^2 + 1 + \frac{1}{x^2} \right) y(x) = 0, & x \in [1, +\infty) \\ y(1) = 1 \end{cases}$$

In this section we are going to solve this equation analytically, since it is possible. This will be useful as reference for the results which will be obtained applying the asymptotic matching method. First, it is possible to write this equation as:

$$\frac{dy}{dx} + \left(\varepsilon x^2 + 1 + \frac{1}{x^2} \right) y(x) = 0$$

then, it is possible to apply the following trick:

$$\frac{dy}{y} = - \left(\varepsilon x^2 + 1 + \frac{1}{x^2} \right) dx$$

Then, it is possible to apply the operator of indefinite integration to both these terms:

$$\int \frac{dy}{y} = - \int \left(\varepsilon x^2 + 1 + \frac{1}{x^2} \right) dx$$

which is:

$$\log y = C - \varepsilon \frac{x^3}{3} - x + \frac{1}{x}$$

where C is an integration constant; now, inverting the logarithm, it is possible to obtain:

$$y(x) = K e^{-\varepsilon \frac{x^3}{3} - x + \frac{1}{x}}$$

where

$$K = e^C$$

Now, it is possible to apply the initial condition:

$$y(1) = K e^{-\frac{\varepsilon}{3}} = 1 \implies K = e^{\frac{\varepsilon}{3}}$$

Therefore, the reference solution which we were looking for is:

$$y(x) = e^{\frac{\varepsilon}{3}} e^{-\varepsilon \frac{x^3}{3} - x + \frac{1}{x}}$$

4.3 Discussion of asymptotic matching

Let us start again from the differential equation, which will be solved by means of different methods:

$$y'(x) + \left(\varepsilon x^2 + 1 + \frac{1}{x^2} \right) y(x) = 0$$

The presence of ε is suggesting us to try a perturbation analysis; to this aim, let us call y_L the unknown of the differential equation with $\varepsilon \rightarrow 0$ (L means Left, as we will discuss):

$$y'_L + \left(1 + \frac{1}{x^2} \right) y_L = 0$$

To obtain this equation it is necessary to have small values of εx^2 ; indeed, if $\varepsilon \rightarrow 0$ but x is very large, this approximation does not make any sense. Therefore, applying this approximation, which leads to the perturbation approach, is correct only for low values of x , which are the left side of the interval; this provides us with an explanation for the name of the solution. Then, this differential equation should be solved with $y_L(1) = 1$.

However, this approximation can not be applied for big values of x ; instead, what is happening for large x is that the $\frac{1}{x^2}$ term becomes irrelevant, therefore:

$$\varepsilon x^2 \gg \frac{1}{x^2}$$

so, it is possible to write the right equation:

$$y'_R + (\varepsilon x^2 + 1)y_R = 0$$

In this case, since the equation is valid only for the right part of the interval, the left boundary condition should not be enforced.

We splitted the problem in two sub-problems, and, independently, we obtained two solutions, valid for each sub-problem. However, in the same interval, the two solutions have to be very close; our objective is to identify the interval where the two solutions have to be matched; this is **asymptotic matching**.

An engineering case study where this situation can occur can be found again in nuclear engineering; for instance, neutrons can be very fast, also at

20 °C, as it can be seen from Boltzmann equation: at these temperatures, neutrons may have a speed of the order of magnitude of 200 m/s, and the size of a nuclear reactor may be 2 meters; therefore, the lifetime of a neutron may be there short, since, after having traveled for a certain path, they can be absorbed or thrown away; the average lifetime of these neutrons may be something like $\tau \sim 10^{-3}$ s.

On the other hand, there are also neutrons given by radioactive processes; in these cases, these neutrons have pregnancy times, which have to be kept into account in their lifetime, and this extends remarkably their life; so, there are two scales for neutrons originated by different processes: μ s for the first kind, and s for the delayed neutrons; this means that we have a **stiff** problem, because it is not possible to apply a trivial time discretization: a reasonable discretization for the fast neutrons may be unreasonable for the delayed ones, because for them, almost only constant quantities are observed; instead, a reasonable discretization for the delayed neutrons may be too rough for the fast ones.

So, how to apply this asymptotic matching method? The first step is to solve the two equations; for the left one, applying once again variable separation:

$$y'_L = - \left(1 + \frac{1}{x^2} \right) y_L \implies \frac{y'_L}{y_L} = \left(-1 + \frac{1}{x^2} \right)$$

so:

$$y_L(x) = Ke^{-x + \frac{1}{x}}$$

and, with the boundary condition, it is possible to show that $K = 1$:

$$y_L(x) = e^{-x + \frac{1}{x}}$$

Then, for what concerns the second equation:

$$y_R(x) = Ae^{-\varepsilon \frac{x^3}{3} - x}$$

where A is the integration coefficient, and it can not be determined by the $x = 1$ boundary condition, since this solution is not valid for $x = 1$. But this means that A is a degree of freedom which can be used in order to let the two functions overlap in the part of the domain where both methods are valid. How to find this region? Well, we have to force the two solutions to have the **same analytical expression**:

$$e^{-x+\frac{1}{x}} = Ae^{-\varepsilon\frac{x^3}{3}-x}$$

if we assume x to be not so small, we get:

$$y_L \sim e^{-x}$$

while, if we put $a = 1$ (this is suggested by the fact that the multiplying constant of the exponential in y_L is 1):

$$y_R = e^{-x}$$

therefore, this can be done if we determine a good matching interval; to this aim, we have to require, for y_R , that, in order to make it equal to e^{-x} :

$$\varepsilon\frac{x^3}{3} \ll x \implies x^2 \ll \frac{1}{\varepsilon}$$

so:

$$x \ll \varepsilon^{-\frac{1}{2}}$$

For the left equation, it is necessary to require that:

$$x \gg \frac{1}{x}$$

so, finally, the interval where matching occurs is:

$$1 \ll x \ll \varepsilon^{-\frac{1}{2}}$$

4.4 Final comments

In order to conclude this chapter on asymptotic matching, some additional comment should be provided.

Firstly, considering the last equation solved, it is possible to ask if

$$1 \ll x \ll \varepsilon^{-\frac{1}{3}}$$

is acceptable as interval. The answer is yes: indeed, $\varepsilon^{-\frac{1}{3}}$ is smaller than $\varepsilon^{-\frac{1}{2}}$, so the bound is satisfied.

What if we choose:

$$\varepsilon^{-\frac{1}{5}} \ll x \ll \varepsilon^{-\frac{1}{3}}$$

In this case, this is not possible, because no information concerning the absolute value of x is provided: only its relationship with ε , so it is possible to obtain low values of x , which can not be accepted.

Another comment is: this explanation on asymptotic matching has been done on a first order approximation; however, if we write y_L as a perturbation series:

$$y_L(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$$

forcing:

$$y_0(1) = 1, y_n(1) = 0, n \geq 1$$

then, we have a higher order approximation of the unknown.

What can be observed applying asymptotic matching with higher order terms is that the validity of the left representation changes, and the same applies to the right representation; what happens is that solutions tend to the reference solution, and the matching coefficient tends to $e^{\frac{\varepsilon}{3}}$.

Chapter 5

Complements to the course

5.1 Eigenvalue problems

5.1.1 Imaging eigenproblems

Which is the philosophy, the physical meaning of an eigenproblem? To try to explain this, it is possible to use some visual and nice example. An eigenvalue problem has the following actors: \hat{v} , which is the operator, defined in a certain domain; y_n , which is the n -th eigenfunction, or eigenvector (if we are considering a finite-dimensional eigenproblem), and λ_n , which is the eigenvalue associated to the n -th eigenfunction.

Let us imagine, in some funny way, that our operator is an **island**: its boundary conditions, which are the boundary which can not be crossed from what lays inside the island, which means the set of things on which our island/operator can operate, is the sea; then, let us assume that we are interested on the population of a certain species, like rabbits, on this island. Our objective is to understand how the island affects the reproduction of rabbits. Indeed, in this island there may be hunters, or wild animals, attacking the poor rabbits; on the other hand, rabbits, depending on their distribution on the island, may be encouraged to reproduce: the island operates on the rabbits distribution, making them encounter or be killed, and after a first application of the island on the initial population, we will have a different amount of rabbits: the second generation.

Let y_0 be the population of the island when we start observing the rabbits distribution; this is the initial condition of the problem. After the first observation, we will observe y_1 , which is the distribution of rabbits generated

by the initial condition, and by the effect of the island:

$$y_1 = \hat{\vartheta}y_0$$

So, y_1 is the second generation of rabbits. The interesting thing is the fact that since y_0 has no information on the island, it is unaware of $\hat{\vartheta}$. Instead, y_1 is **caused** by both y_0 and $\hat{\vartheta}$, so it has some information: indeed, the island may cause propagation, killing events or other phenomena. But y_1 does not know the entire information of the island: indeed, it knows only the effect of $\hat{\vartheta}$ on the specific y_0 distribution; moreover, the effect of $\hat{\vartheta}$ on y_0 is not “complete”: we just observed at a certain time the distribution y_1 , without being sure that balance has been achieved. Then it is possible to go on, to observe another time the distribution of rabbits, and to see:

$$y_2 = \hat{\vartheta}y_1$$

and so on.

Now, let us consider the integral of y_0 , called $\langle y_0 \rangle$, and the same for y_1 , $\langle y_1 \rangle$, then, let us compute the ratio R_1 as:

$$R_1 = \frac{\langle y_1 \rangle}{\langle y_0 \rangle}$$

this has some dependence on $\hat{\vartheta}$, and, depending on it, it can be bigger or lower than 1: if the island is encouraging reproduction, the number of rabbits will increase while, if the island is discouraging reproduction, the number of rabbits will decrease. In general:

$$R_n = \frac{\langle y_n \rangle}{\langle y_{n-1} \rangle}$$

is the ratio of the rabbits at the n -th observation to the one at the $n - 1$ -th observation. Newer observations correspond to longer times, and rabbits, after a certain time, find their balance with the island: they learn to live with the island, and the reproduction phenomenon tends to become always the same. This means that, for a high n (which means a generation very far from the initial one), this ratio tends to become constant:

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{\langle y_n \rangle}{\langle y_{n-1} \rangle} = \lambda$$

indeed, if this ratio tends to be constant, we have that:

$$y_n = \hat{\vartheta}y_{n-1}$$

but:

$$y_n = R_n y_{n-1}$$

so:

$$R_n y_{n-1} = \hat{\vartheta}y_{n-1}$$

if $n \rightarrow \infty$, it is possible to confuse n and $n - 1$, and this means that R_n becomes constant and equal to this λ :

$$\hat{\vartheta}y_{n-1} = \lambda y_{n-1}$$

and this λ , as it is written here, is the **eigenvalue of the operator**.

This rabbits story helped us to understand (I hope so!) the meaning of λ : it is basically a parameter which provides information on the asymptotic state of the system, so on the **steady state**. Indeed, λ is fairly related to initial conditions (not directly, because, for $n = 1$, R_1 is very different from R_n , for big n); instead, it is related to the physics of the system. This is why it is called **eigenvalue**: this is a **proper value** of the system, related to the homogeneous part, not on the excitation! It is related to how the system behaves, once it is left by itself.

A generic operator equation has an expression like:

$$(\hat{\vartheta} - \lambda \hat{I} = y(x) = S(x)$$

where:

$$y(x) = y_h(x) + y_p(x)$$

and $y_h(x)$ is the homogeneous solution, obtained for $S(x) = 0$ (the **proper solution**), and $y_p(x)$ is the particular solution, caused by the source. Since y_h is a solution of:

$$(\hat{\vartheta} - \lambda \hat{I} = y_h(x) = 0$$

then it is possible to write, just for it:

$$\hat{\vartheta}y_h = \lambda y_h$$

usually there is not only a solution to this problem, but there are several values $\{\lambda_i\}$; this set can be finite, numerable, or also continuous. This is related to the spectrum of the operator $\hat{\vartheta}$. The eigenfunction y_i is a solution which lives without any excitation from outside: a proper state of the system.

Knowing the spectrum of the operator means knowing everything of it; indeed, mathematicians have shown that the set of eigenvectors (or eigenfunctions) is a basis for the space where the operator is defined. Moreover, since every vector of the space can be written as a linear combination of other vectors belonging to it, it is possible to write every solution of the operator equation as a linear combination of the eigenfunctions!

This, in \mathbb{R}^n , is:

$$\underline{y} = \sum_{i=1}^N c_i \underline{u}_i$$

where \underline{u}_i are the eigenvectors of our space. Instead, with a continuous spectrum:

$$f(x) = \sum_i c_i y_i(x) \xrightarrow{\text{continuity}} \int_{\mathcal{D}} g(\xi) y(\xi, x) d\xi$$

One property of eigenvectors, in many problems, is the fact that they are orthogonal (and, since the solution of an eigenvalue problem provides information only on the “orientation” of the vectors, it is possible to normalize them and make them orthonormal); however, this occurs if the operator is selfadjoint, which means that it equals its adjoint. On the other hand, if our problem has no orthogonal eigenvectors, it is possible to use the trick which we adopted some chapters ago, and to build the **adjoint problem**:

$$\hat{\vartheta}^\dagger y^\dagger = \lambda^\dagger y^\dagger$$

the solutions of the adjoint problem are biorthogonal to the ones of the direct problem, and so projection can be applied to them, in order to have a smart basis on which calculate the coefficients; smart means that, if all the projections are zero except for one contribution, each coefficient can be calculated as a single projection: indeed, all other contributions equal zero! This is why orthogonal bases are so attractive: each contribution depend

only on a single term, so it is possible to improve the approximation, by adding new terms, without needing to modify the previous coefficients.

5.1.2 Fourier analysis and heat equation

Even if the names “eigenvalue” or “eigenvector” come from the 19th century, the inventor of the idea behind eigenproblems was Fourier. At that time, one of the most important problems which had to be studied was heat conduction, which was shown to satisfy the so-called **heat equation**:

$$\frac{\partial f(\underline{r}, t)}{\partial t} = a\nabla^2 f(\underline{r}, t) + bf(\underline{r}, t) + S(\underline{r}, t)$$

where ∇^2 is the Laplace operator (operating only on spatial coordinates), and the solution f depends on both space \underline{r} and time t . Fourier, as well as other mathematicians, didn't like the presence of the ∇^2 operator, because it complicates the expression very much. But Fourier had a very genial idea: he knew that there were functions y_n , called **harmonic functions**, which were the solutions of the homogeneous Laplace equation, *i.e.*:

$$\nabla^2 y_n(\underline{r}) = \lambda_n y_n(\underline{r})$$

to make a simple example of these functions, in one dimension:

$$\frac{d^2}{dx^2} \cos(mx) = -m^2 \cos(mx)$$

therefore, for this case, $\lambda = -m^2$. Extending this idea to the three-dimensional functions, the brilliant idea of Fourier was to represent the solution of the heat equation $f(\underline{r}, t)$ as:

$$f(\underline{r}, t) = \sum_{n=1}^{\infty} c_n(t) y_n(\underline{r})$$

so, he applied a variable separation between time and space, and he expanded the spatial part of the solution as a sum of harmonic functions. Why is this so useful? Well, let us substitute this expression in the heat equation:

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} c_n(t) y_n(\underline{r}) = a\nabla^2 \sum_{n=1}^{\infty} c_n(t) y_n(\underline{r}) + b \sum_{n=1}^{\infty} c_n(t) y_n(\underline{r}) + S(\underline{r}, t)$$

applying linearity, this equation becomes:

$$\sum_{n=1}^{\infty} y_n(\underline{r}) \frac{\partial c_n(t)}{\partial t} = a \sum_{n=1}^{\infty} c_n(t) \nabla^2 y_n(\underline{r}) + b \sum_{n=1}^{\infty} c_n(t) y_n(\underline{r}) + S(\underline{r}, t)$$

but, since $y_n(\underline{r})$ are harmonic functions, it is possible to substitute the eigenvalue relationship:

$$\sum_{n=1}^{\infty} y_n(\underline{r}) \frac{\partial c_n(t)}{\partial t} = a \sum_{n=1}^{\infty} c_n(t) \lambda_n y_n(\underline{r}) + b \sum_{n=1}^{\infty} c_n(t) y_n(\underline{r}) + S(\underline{r}, t)$$

this means that we replaced the ∇^2 operator with λ_n !

Now, it is possible to remember that it is possible to obtain either an orthogonal basis with y_n , or a biorthogonal basis with the adjoint Laplace equation, and so it is possible to project on the orthogonal or biorthogonal terms (multiply by means of inner product) this equation, obtaining:

$$\left\langle y_m, \sum_{n=1}^{\infty} y_n(\underline{r}) \frac{\partial c_n(t)}{\partial t} \right\rangle = \left\langle y_m, a \sum_{n=1}^{\infty} c_n(t) \lambda_n y_n(\underline{r}) + b \sum_{n=1}^{\infty} c_n(t) y_n(\underline{r}) + S(\underline{r}, t) \right\rangle, \quad \forall m$$

which becomes:

$$\begin{aligned} \sum_{n=1}^{\infty} \langle y_m(\underline{r}), y_n(\underline{r}) \rangle \frac{\partial c_n(t)}{\partial t} &= a \sum_{n=1}^{\infty} c_n(t) \lambda_n \langle y_m(\underline{r}), y_n(\underline{r}) \rangle + \dots \\ &+ b \sum_{n=1}^{\infty} c_n(t) \langle y_m(\underline{r}), y_n(\underline{r}) \rangle + \langle y_m(\underline{r}), S(\underline{r}, t) \rangle, \quad \forall m \end{aligned}$$

but these inner products are constants, which depend on the normalization:

$$\langle y_m(\underline{r}), y_n(\underline{r}) \rangle = \kappa_{mn} \delta_{mn}$$

so, since all these constants are the same on the same equation, we obtain:

$$\frac{\partial c_n(t)}{\partial t} = a c_n(t) \lambda_n + b c_n(t) + S_m(t), \quad \forall m$$

where the source depends only on t , since the dependence on \underline{r} has been removed by integrating on it (inner product):

$$S_m(t) = \langle y_m(\underline{r}), S(\underline{r}, t) \rangle$$

This is a set of **ordinary differential equations**, which are very easy to be solved. So, by exploiting the eigenvalue idea, we reduced a parabolic partial differential equation to a set of ordinary differential equations. This is the very same idea applied by Marcuvitz and Schwinger for the modal analysis in waveguide theory.

5.2 Computation of integrals

In this section we are going to calculate, using perturbative methods, an indefinite integral. The problem under analysis is:

$$F(\varepsilon) = \int_0^{+\infty} e^{-t-\frac{\varepsilon}{t}} dt$$

The integral F is a function of the parameter ε ; in this analysis, we are interested in **small values of ε** . So, let us consider the zero order approximation:

$$F(0) = \int_0^{+\infty} e^{-t} dt = 1$$

The idea which we can try to exploit is to transform this problem in an **initial value problem**: sometimes, when working on perturbation analysis, we should look at an equivalent initial value problem; how can we do this? Well, the idea is quite simple: it is possible to differentiate this integral with respect to the parameter ε :

$$F'(\varepsilon) = \int_0^{+\infty} \left(-\frac{1}{t}\right) e^{-t-\frac{\varepsilon}{t}} dt$$

where $F(0) = 1$. The problem now is that $F(\varepsilon)$ was a converging integral while the last one, in $F'(\varepsilon)$, is singular; the differentiation operator introduced singularities. It is possible to try to differentiate another time with respect to ε , and what we obtain is:

$$F''(\varepsilon) = \int_0^{+\infty} \frac{1}{t^2} e^{-t-\frac{\varepsilon}{t}} dt$$

so:

$$F''(\varepsilon) = \frac{1}{\varepsilon}F(\varepsilon =$$

So, we built a differential problem starting from an integral problem:

$$\begin{cases} F''(\varepsilon) = \frac{1}{\varepsilon}F(\varepsilon) \\ F(0) = 1 \end{cases}$$

It is possible to apply a change of variables, in order to obtain:

$$F(\varepsilon) = 2\sqrt{\varepsilon}K_1(2\sqrt{\varepsilon})$$

where K_1 is the Kelvin function: the modified Bessel function of second kind. This means that this program has a reference solution, which can be used to perform benchmark. Moreover, the expression of the differential problem is very suitable for perturbation analysis.

It is possible to write:

$$2\sqrt{\varepsilon}K_1(2\sqrt{\varepsilon}) = 1 + \varepsilon \log \varepsilon + \varepsilon (2\sqrt{\varepsilon} - 1) + \frac{1}{2}\varepsilon^2 \log \varepsilon$$

This is a singular perturbation problem, because there is no way to produce a Taylor series for the perturbation; so, here perturbation analysis can be carried out only by means of an asymptotic matching method. So, let us consider quickly the application of the asymptotic matching method, in order to find the intervals of validity:

- for small ε values, large t values, the term $\frac{\varepsilon}{t}$ becomes negligible, and so $-t$ in the exponential becomes dominant:

$$t \ll 1, e^{-t}e^{-\frac{\varepsilon}{t}} \implies e^{-t} \simeq 1 - t + \frac{t^2}{2} + \dots$$

- the second case is $t \gg \varepsilon$; in this case:

$$e^{-t}e^{-\frac{\varepsilon}{t}} \simeq e^{-t} \left(1 - \frac{\varepsilon}{t} + \frac{\varepsilon}{2t^2} - \dots \right)$$

this is the MacLaurin expansion of $\frac{1}{t}$ for $t \rightarrow \infty$.

This interval can not be determined a priori; indeed, we have two expansions: one for low t values, one for $t \rightarrow \infty$. This means that it is necessary to use two different representations of the integral; in other words, it is necessary to break the integration domain $[0, +\infty)$ in $[0, \delta] + [\delta, +\infty)$:

$$F(\varepsilon) = \int_0^\delta e^{-t-\frac{\varepsilon}{t}} dt + \int_\delta^{+\infty} e^{-t-\frac{\varepsilon}{t}} dt$$

them it is possible to apply perturbation analysis on these integrals; however, the problem is that it is necessary to determine the value of δ as a function of ε : indeed, fixed ε , the point is moving.

5.3 Reducing a model with perturbation analysis

In this section we are going to show briefly how a model can be simplified by means of perturbation analysis, exploiting some physical considerations.

Let us consider, as an example, the Boltzmann transport equation in one dimension:

$$\mu \frac{\partial \varphi}{\partial x} + \sigma(x) \varphi(x, \mu) = \frac{1}{2} \sigma_s \int_{-1}^{+1} \varphi(x, \mu') d\mu' + S(x, \mu)$$

This is an integro-differential equation, where $\mu = \cos \vartheta$, and ϑ is a certain angle; our objective is to change the structure of this equation working with perturbation methods; for instance, it is possible to assume the presence of **rarified electrons**, in order to apply some simplification to the model.

Let us assume that, given σ_a the absorption parameter, given σ the total collisions parameter, and σ_s the scattering collision parameters, the absorbed particles are all the particles which are not scattered, therefore:

$$\sigma_a = \sigma - \sigma_s \implies \sigma_s = \sigma - \sigma_a$$

If we want for instance to obtain a model where few particles are absorbed, instead of σ_a we can consider $\varepsilon \sigma_a$, where $\varepsilon \rightarrow 1$; then, we can desire small absorption, but a lot of collisions; in this case we divide σ by our ε instead than multiplying times it:

$$\sigma_s = \frac{1}{\varepsilon} \sigma - \varepsilon \sigma_a$$

This comes from physical considerations: small absorption, and high collision rate.

For what concerns the source, we are looking for particle density, so S is the source; close to the source, we have anisotropic functions, but far away from the sources, particles are scattering in any direction; if we want to consider to be far from the source, $S \rightarrow \varepsilon S$. Using these last two approximations, it is possible to re-write the equation:

$$\mu \frac{\partial \varphi(x, \mu)}{\partial x} + \frac{\sigma}{\varepsilon} \varphi(x, \mu) = \frac{1}{2} \left(\frac{\sigma}{\varepsilon} - \sigma_a \varepsilon \right) + \varepsilon S(x, \mu)$$

Now, if we define:

$$\Phi(x) = \int_{-1}^{+1} \varphi(x, \mu') d\mu'$$

Then, we can consider the following perturbation expansion for φ and Φ :

$$\begin{cases} \varphi(x, \mu) = \varphi_0(x, \mu) + \varepsilon \varphi_1(x, \mu) \\ \Phi(x) = \Phi_0(x) + \varepsilon \Phi_1(x) \end{cases}$$

If we take these two expressions, we substitute them in the equation, we drop high order terms, can we foresee what is going to happen? Well, the leading terms of the equations are the ones which are $O(\varepsilon^{-1})$, which require a singular perturbation analysis; we will get relationships for φ , for Φ , and we can focus on Φ_0 ; what will be obtained is an independent equation of Φ_0 :

$$\frac{d}{dx} \left(\frac{1}{3\sigma(x)} \frac{d\Phi_0(x)}{dx} - \sigma_a(x) \Phi_0(x) + S_0 \right) = 0$$

this is a diffusion equation, where $\sigma(x)$ are the diffusion coefficients; therefore, we quickly proved that the diffusion model can be derived starting from the transport theory, stopping the perturbation series to the first term. This is procedure of model reduction.