Coherent states: time evolution

Given a state $|\tilde{x}_0\rangle$, what is $|\tilde{x}_0, t\rangle$? At a later time, t ?
Now we know Heisenberg picture, so studying the expectation values is reasonable! In particular, we will start studying for a Schrödinger operator $\hat{A}$, the "sandwich"

$$\langle \tilde{x}_0, t | \hat{A} | \tilde{x}_0, t \rangle \equiv \langle A \rangle_t$$

$$= \langle \tilde{x}_0 | \hat{A}_H | \tilde{x}_0 \rangle \quad \left(\equiv \langle 0 | \hat{T}^\dagger_{x_0} \hat{A}_H \hat{T}_{x_0} | 0 \rangle \right)$$

Let's try to do something for an example! What is $\langle \hat{x} \rangle_{\tilde{x}_0}(t)$?

$$= \langle \tilde{x}_0 | \left(\hat{x} \cos(\omega t) + \frac{\hat{p}}{m\omega} \sin(\omega t) \right) | \tilde{x}_0 \rangle \; : \text{ last lecture, } \hat{x}_H(t) !$$

but $\langle \tilde{x}_0 | \hat{p} | \tilde{x}_0 \rangle = 0$, so

$$= \langle \tilde{x}_0 | \hat{x} \cos\omega t | \tilde{x}_0 \rangle = x_0 \cos\omega t \; .$$ Just as we said! Back and forth!

How about

$\langle \hat{p} \rangle_{\tilde{x}_0}(t)$? Similar! If it oscillates, it'll better be wrong

$$\hookrightarrow \langle \tilde{x}_0 | \hat{p} \cos\omega t - m\omega \hat{x} \sin\omega t | \tilde{x}_0 \rangle = -m\omega x_0 \sin\omega t \left(= m \frac{d}{dt} \langle \hat{x} \rangle (t) \right)$$

One is the derivative of the other, very classical!

Another very important thing we want to compute, very important, is: uncertainty! We know that the ground state is characterized by a "minimum uncertainty packet", so, is it real that the shape is preserved during motion? Which we can verify by computing Δx and Δp in time? We don't want them to change! The property of coherent states, which now we want to verify, is that Δx and Δp do NOT change in time, as well as the shape of the states. So: how do $\Delta x(t)$, $\Delta p(t)$ behave?

Let's use Heisenberg picture.

$$(\delta x)^2(t) = \langle \tilde{x}_0, t | \hat{x}^2 | x_0, t \rangle - \langle \tilde{x}_0, t | \hat{x} | x_0, t \rangle^2 .$$

This is the uncertainty (see old lectures ;-)

By turning this in Heisenberg,

$$\hookrightarrow = \langle \tilde{x}_0 | \hat{x}_H^2(t) | \tilde{x}_0 \rangle - x_0^2 \cos^2 \omega t$$ (just computed.)

The complex term is the first:

$$\langle \tilde{x}_0 | x_H^2(t) | \tilde{x}_0 \rangle = \langle \tilde{x}_0 | \hat{x}^2 \cos^2 \omega t + \frac{\hat{p}^2}{m^2\omega^2} \sin^2 \omega t + \frac{1}{m\omega} \cos \omega t \sin \omega t (\hat{x}\hat{p} + \hat{p}\hat{x}) | \tilde{x}_0 \rangle ,$$

But, in the exercises suggested before, we revealed (and asked to show) that

$$\langle \tilde{x}_0 | (\hat{x}\hat{p} + \hat{p}\hat{x}) | \tilde{x}_0 \rangle = 0 .$$ (p. 31 lecture notes)

Instead,

$$\langle \tilde{x}_0 | \hat{x}^2 | \tilde{x}_0 \rangle = x_0^2 + \frac{\hbar}{2m\omega}$$

and

$$\langle \tilde{x}_0 | \hat{p}^2 | \tilde{x}_0 \rangle = \frac{m\hbar\omega}{2} .$$

So that we obtain

$$= \left(x_0^2 + \frac{\hbar}{2m\omega} \right) \cos^2 \omega t + \frac{m\hbar\omega}{2m^2\omega^2} \sin^2 \omega t - x_0^2 \cos^2 \omega t$$

$$= x_0^2 \cancel{\cos^2 \omega t} + \frac{\hbar}{2m\omega} \cos^2 \omega t + \frac{\hbar}{2m\omega} \sin^2 \omega t - x_0^2 \cancel{\cos^2 \omega t} .$$

$$(\delta x)^2(t) = \frac{\hbar}{2m\omega} .$$ constant in time!!!

and it is the original uncertainty of the ground state!

From lecture notes it is possible to discover that $(\delta p)^2$ is constant in time as well ;-) (p. 32):

$$(\delta p)^2(t) = \frac{m\hbar\omega}{2} .$$

So,

$$\left[(\delta p)(t) \right] \left[(\delta x)(t) \right] = \frac{\hbar}{2}$$;-)

Now, the name "coherent" should make sense: it refers to quantum states that have about the energy of states we're familiar with, and that, moving (classically!), don't change shape!

The next step is, then, to study these states in the energy basis. Saying that a superposition of states with different energies have these properties is incredible! Now we're going to look into this. And this will allow us to generalize the definition of coherent states!

Energy basis 46 min

After defining the coherent states $|\tilde{x}_0\rangle$ (that's not just time now!),

$$|\tilde{x}_0\rangle = \exp\left(-i \frac{\hat{p} x_0}{\hbar} \right) |0\rangle .$$

When discussing the harmonic oscillator, often it is possible to find a famous length scale, d_0, (see 8.04x lectures), such that:

$$d_0^2 = \frac{\hbar}{m\omega} .$$

This is essentially the uncertainty of the position of the ground state, up to a $\sqrt{2}$. Based on this, we can then write $\hat{p}$ as:

$$|\tilde{x}_0\rangle = \exp\left(-i \frac{\hat{p} x_0}{\hbar} \right) |0\rangle = \exp\left(\frac{x_0}{\sqrt{2} d_0} (\hat{a}^\dagger - \hat{a}) \right) |0\rangle$$ (it can be shown by performing the computations.) (1)

Notice that $\hat{a}^\dagger$ and $\hat{a}$ have no units!

Now: how does this figure out in the energy basis? Well, to find out, we have to work on the exponential! According to Baker-Campbell-Hausdorff, we have that

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]}$$

and this stops there if and only if $[[X,Y],X] = 0$, $[[X,Y],Y] = 0$!

This expression is useful because it allows us to split the creation and annihilation operators in separate exponentials! So as we can write

$$|\tilde{x}_0\rangle = \exp\Big(\underbrace{\frac{x_0}{\sqrt{2}\,d_0}\hat{a}^\dagger}_{X} - \underbrace{\frac{x_0}{\sqrt{2}\,d_0}\hat{a}}_{Y}\Big)$$, we have:

(notice that it is useful to have annihilation at right and creation at left!) (in fact, annihilation close to vacuum $|0\rangle$ is simple: it is 0! instead, putting $\hat{a}^\dagger|0\rangle$ complicates the calculations!!!)

So, we have:

$$|\tilde{x}_0\rangle = \exp\Big(\frac{x_0}{\sqrt{2}\,d_0}\hat{a}^\dagger\Big)\exp\Big(-\frac{x_0}{\sqrt{2}\,d_0}\hat{a}\Big)\exp\Big(-\frac{1}{2}\frac{x_0^2}{2\,d_0^2}\Big[\hat{a}^\dagger,\hat{a}\Big](-1)\Big)|0\rangle$$

But, $\Big[\hat{a}^\dagger,\hat{a}\Big]=1$! It is a NUMBER! This is crucial!! Otherwise, it would not so easily be true that it would commute with $\hat{a}$ or $\hat{a}^\dagger$! So, this equals:

$$|\tilde{x}_0\rangle = \exp\Big(\frac{x_0}{\sqrt{2}\,d_0}\hat{a}^\dagger\Big)\exp\Big(-\frac{x_0}{\sqrt{2}\,d_0}\hat{a}\Big)\exp\Big(-\frac{1}{4}\frac{x_0^2}{d_0^2}\Big)|0\rangle$$

So, the last exponential is just a number, and it doesn't affect the operator. Then, we have the exponential of two annihilator operators; to study how it acts on the vacuum $|0\rangle$, we have to expand it in power series:

$$\exp\Big(-\frac{x_0}{\sqrt{2}\,d_0}\hat{a}\Big) = \mathbb{1} - \frac{x_0}{\sqrt{2}\,d_0}\hat{a} + \frac{x_0^2}{\sqrt{2}\,d_0}\hat{a}^2 + \dots$$

and all the operators involving $\hat{a}|0\rangle$ are equal to 0. So, just $\mathbb{1}$ remains! So, we have:

$$|\tilde{x}_0\rangle = \underbrace{\exp\Big(-\frac{1}{4}\frac{x_0^2}{d_0^2}\Big)}_{\text{constant}}\exp\Big(\frac{x_0}{\sqrt{2}\,d_0}\hat{a}^\dagger\Big)|0\rangle \qquad !!!$$

They are not quite energy eigenstates, but we're not so far from them! In fact, this is:

$$= \exp\Big(-\frac{1}{4}\frac{x_0^2}{d_0^2}\Big)\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{x_0}{\sqrt{2}\,d_0}\Big)^n(\hat{a}^\dagger)^n|0\rangle. \quad (*)$$

Now, a question: what was the n-th energy eigenstate of the harmonic oscillator? Well, it was...

$$\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$ So, not so different! But,

$$(*) = \exp\Big(-\frac{1}{4}\frac{x_0^2}{d_0^2}\Big)\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\Big(\frac{x_0}{\sqrt{2}\,d_0}\Big)^n|n\rangle =$$

A little messy but not so bad!

$$= \sum_n c_n|n\rangle$$, where we know the $\{c_n\}$.

This is a very precise superposition of energy eigenstates. We can also write the probability to find the coherent state in the n-th energy eigenstate, c_n^2, as:

c_n^2 = probability for $|\tilde{x}_0\rangle$ to be $|n\rangle$

$$= \exp\Big(-\frac{1}{2}\frac{x_0^2}{d_0^2}\Big)\frac{1}{n!}\Big(\frac{x_0^2}{2\,d_0^2}\Big)^n.$$

If we define

$$\lambda = \frac{x_0^2}{2d_0^2}$$, we have

$$c_n^2 = \exp(-\lambda)\frac{\lambda^n}{n!}$$

What is really strange about this, is that this is the so-called Poisson distribution. There's something about these coherent states for which they are "Poisson distributed" over energy eigenstates.

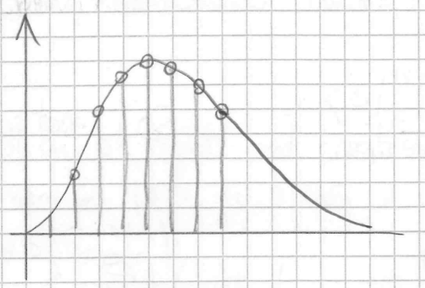
Notice that

$$\sum_{n=0}^{\infty}c_n^2 = 1.$$

In fact,

$$\sum_{n=0}^{\infty} c_n^2 = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda}\left(e^{\lambda}\right) = 1$$

So, something like this!



One more remark about this.

What is the most probable n?

So, which is the expectation value of $\langle n \rangle$?

Let us think of this probabilistically, so just like c_n^2 is a probability distribution then

$$\langle n \rangle = \sum_{n=0}^{\infty} n c_n^2 = \sum_{n=0}^{\infty} n\, e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=1}^{\infty} n \frac{\lambda^n}{n!} =$$

$$= e^{-\lambda} \lambda \frac{d}{d\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} \lambda \frac{d}{d\lambda} e^{\lambda} = \lambda.$$

This λ is the level which will most likely be. So, since

$$\lambda = \frac{x_0^2}{2 d_0^2}.$$

which means, if $x_0 = 1000\, d_0$ for instance, that the coherent state has been obtained by moving the energy eigenstates away "1000 times" the uncertainty d_0, $n \approx 10^6$!

Now. Let us assume to compute, instead of $\langle n \rangle$, the

$$\langle \tilde{x}_0 | \hat{N} | \tilde{x}_0 \rangle.$$

$\hat{N}$ is the number operator, the star of the harmonic oscillator!

$$\hookrightarrow = \sum_n \sum_m c_m^* \langle m | \hat{N} | n \rangle c_n \quad . \text{ But } \hat{N}|n\rangle = n|n\rangle, \text{ so,}$$

$$\hookrightarrow = \sum_n \sum_m c_m^* \langle m|n\rangle c_n = \sum_n \sum_m c_m c_n \delta_{mn} c_n = \sum_n n c_n^2,$$

which is what we just calculated "probabilistically" :-)

Another very interesting calculation that we could now perform is the uncertainty in the energy. Since: which is the ΔE associated to a coherent state? Are coherent states well localized in energy?

In the lecture notes it is shown that...

$$(\Delta E)_{|\tilde{x}_0\rangle} = \hbar\omega \frac{x_0}{\sqrt{2}\, d_0}$$

So, what is this "classically-looking coherent state"? It is a state with $x_0 \gg d_0$. By this way, $\frac{x_0}{d\sqrt{2}}$ is a large number, but then the uncertainty ΔE results to be big compared to $\hbar\omega$, the spacing of the energy eigenstates. So, if ΔE is big a lot of energy eigenstates is excited: $\hbar\omega$ is the spacing, so $\Delta E \gg \hbar\omega$ means that several states are excited. In this sense.

But, $\frac{\langle E \rangle}{\Delta E} \frac{\frac{1}{2} m\omega^2 x_0^2}{\frac{x_0}{d\sqrt{2}} \hbar\omega} = \frac{x_0}{\sqrt{2}\, d_0}$

This means that this state contains an uncertainty that embraces several states, BUT, this uncertainty is smaller than $\langle E \rangle$ of the same amount: the state is localized in energy, even if several eigenstates are excited!

Now. Let us generalize our coherent states, to something more flexible.

Inspired by the expression (1), we can write the following operator:

$$|\alpha\rangle \overset{\Delta}{=} \hat{D}(\alpha)|\phi\rangle \quad , \text{ where } \hat{D} \text{ is an unitary operator, } \alpha \in \mathbb{C} \text{ is a complex number}$$

$$= \exp\left(\alpha \hat{a}^{\dagger} - \alpha^* \hat{a}\right)|\phi\rangle$$

With complex α, this is more complicated than $\alpha = \frac{x_0}{\sqrt{2} d_0}$. Still, $\hat{D}(\alpha)$ is unitary, and preserves the norm!

To figure out what this operator does, we should calculate

its action on something. For instance, what is

$\hat{a}|\alpha\rangle$?

$$\hat{a}|\alpha\rangle = \hat{a}\exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle =$$

$$= \left[\hat{a},\ \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)\right]|0\rangle.$$

This is not trivial, but: since $\hat{a}|0\rangle = 0$, we have that

$$\left[\hat{a},\ \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)\right]|0\rangle = \hat{a}\exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle - \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)\underbrace{\hat{a}|0\rangle}_{=0}$$

So, since

$$\left[A,\exp(B)\right] = \left[A,B\right]\exp(B), \text{ if } \left[\left[A,B\right],B\right] = 0, \text{ so,}$$

$$\hookrightarrow \left[\hat{a},\exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)\right] = \left[\hat{a},\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right]\exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$$

$$= \alpha\exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right) = \alpha|\alpha\rangle \quad !!!$$

So,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

We have diagonalized a non-Hermitian operator! :O
Actually, it could happen! But then, states corresponding to different eigenvalues are not orthogonal, do not constitute a complete set of states, and so on. Still, quite remarkable!

This characterizes the coherent states in a very nice way: they are the eigenstates of the annihilation operator!

We knew what happened for real α: a complex α gives to the original coherent state some momentum. In fact, for position, we have

$$\langle\alpha|\hat{x}|\alpha\rangle = \frac{d}{\sqrt{2}}\langle\alpha|\hat{a}+\hat{a}^\dagger|\alpha\rangle = \frac{d}{\sqrt{2}}\left(\langle\alpha|\underbrace{\hat{a}|\alpha\rangle}_{\alpha|\alpha\rangle} + \underbrace{\langle\alpha|\hat{a}^\dagger}_{\langle\alpha|\alpha^*}|\alpha\rangle\right) =$$

$$= \frac{d}{\sqrt{2}}\left(\alpha+\alpha^*\right) = \sqrt{2}\,d\,\mathrm{Re}\{\alpha\}$$

The REAL part is related to the exp. value of $\hat{x}$!

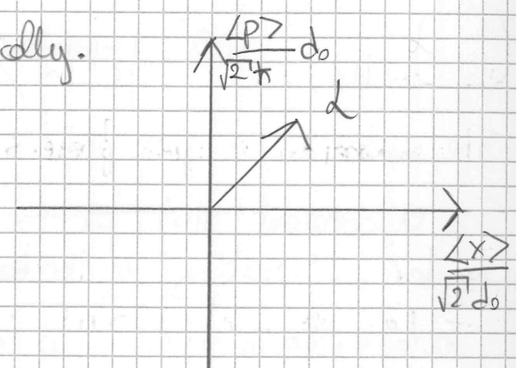
Similarly,

$$\langle\hat{p}\rangle = \langle\alpha|\hat{p}|\alpha\rangle = \frac{\sqrt{2}\hbar}{d}\,\mathrm{Im}\{\alpha\} \qquad \text{(skipping steps...)}$$

These are much more general: $\mathrm{Re}\{\alpha\}$ translates the state, and $\mathrm{Im}\{\alpha\}$ "kicks" it - gives it a momentum!

These states can be also described geometrically.

Let us report the complex number α in a complex plane. This means that the abscissa can be interpreted as the expectation value of the position, and the ordinates of the momentum.



Let this arrow identify our state at $t=0$.
What happens for later times?

Well,

$$|\alpha,t\rangle = \underbrace{e^{-i\frac{\hat{H}t}{\hbar}}\ e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\ e^{i\frac{\hat{H}t}{\hbar}}}\ e^{-i\frac{\hat{H}t}{\hbar}}|0\rangle$$

But the right part is simple:

$$e^{-i\frac{\hat{H}t}{\hbar}}|0\rangle = e^{-i\frac{\omega t}{2}}|0\rangle$$

Instead, the part above the curly bracket is pretty much equal to the Heisenberg operator, just with a wrong sign. Actually, this is simply the operator at $-t$ instead than $+t$:

$$e^{-i\frac{\hat{H}t}{\hbar}}\ e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\ e^{+i\frac{\hat{H}t}{\hbar}} = e^{\alpha\hat{a}_H^\dagger(t) - \alpha^*\hat{a}_H(-t)}$$

and we know $\hat{a}_H(t)$, $\hat{a}_H^\dagger(t)$, so:

$$\hat{a}_H(t) = e^{-i\omega t}\hat{a}$$

$$\hat{a}_H^\dagger(t) = e^{+i\omega t}\hat{a}^\dagger$$

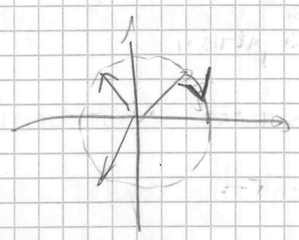
and we can just substitute with $-t$!

$$\Rightarrow |\alpha,t\rangle = e^{\alpha e^{-i\omega t}\hat{a}^\dagger - \alpha^* e^{+i\omega t}\hat{a}} e^{-\frac{i\omega t}{2}\hat{n}} |0\rangle$$

This means that, instead of α, we have $\alpha e^{-i\omega t}$!

So, a rotation on the complex plane!

$$\hookrightarrow |\alpha,t\rangle = e^{-i\frac{\omega t}{2}} \left| e^{-i\omega t}\alpha \right\rangle$$



The number α is just phase-shifted!

So, a coherent state, at the end, can be visualized as a complex number "rotating" in the complex plane!

Lecture 15: Quantum Dynamics (continued)

Now we will study ideas based on coherent states: squeezed states, and photon states.

Before starting with squeezed states, let us develop this a little more.

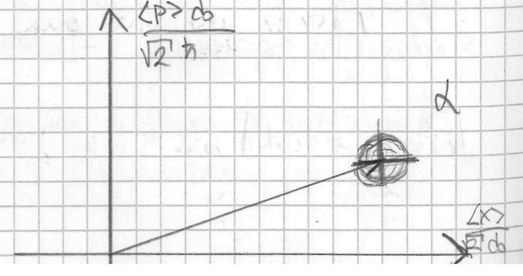
We said that α is a complex number, such that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

However, $|\alpha\rangle$ is not a position, nor momentum, nor energy eigenstate: it is an eigenstate of the destruction operator! So, it has all kinds of uncertainties!

So, $\mathrm{Re}\{\alpha\} = \frac{\langle x\rangle}{\sqrt{2}x_0}$, $\mathrm{Im}\{\alpha\} = \frac{\langle p\rangle x_0}{\sqrt{2}\hbar}$

But, since we have uncertainties, position is not well defined: it is not necessarily EQUAL to the expectation value.



We have this "blob": α is around this blob; the expectation values are of course well defined, but $(\Delta x)_{|\alpha\rangle} \neq 0$, $(\Delta p)_{|\alpha\rangle} \neq 0$

Let us relate this with the idea of waves. Assume that we have a wave with energy E, a light wave. So, something like

$A\cos(\omega t)$

We discussed about energy-time uncertainty, $\Delta E \Delta t$. Let us talk about this now, firstly with some "not-so-rigorous" approach.

Given our wave, the phase, $\phi = \omega t$, can intuitively tell us that

$$\Delta\phi = \omega \Delta t \Rightarrow \frac{\Delta\phi}{\omega} = \Delta t$$

Being a light wave, our wave has some number of photons; so

$E = N\hbar\omega$, where N is the number of photons,

$$\Delta E = \Delta N \hbar\omega$$

Let us substitute in $\Delta E \Delta t \approx \frac{\hbar}{2}$:

$$\Rightarrow \Delta N \cancel{\hbar\omega} \frac{\Delta\phi}{\cancel{\omega}} \approx \frac{\hbar}{2} \Rightarrow \boxed{\Delta N \Delta\phi \approx \frac{1}{2}}$$

This relation is kinda serious: if we have a wave in quantum optics, it is said that the uncertainty in the number of photons and the uncertainty in the coherence of these photons (if they are out of phase they are not coherent), satisfy this relationship.

Let us do some more precise derivation based on coherent states: this was quite intuitive but not rigorous.

Firstly, a preliminary calculation. In the coherent state $|\alpha\rangle$, what is the expectation value of the number operator $\hat{N}$?

$$\langle\hat{N}\rangle_{|\alpha\rangle} = \langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2\langle\alpha|\alpha\rangle = |\alpha|^2!$$

($\langle\alpha|\alpha^*$ and $\alpha|\alpha\rangle$)

That means, that the expectation value of the number operator is the length of the vector in the last page picture.

How about $\langle\hat{N}^2\rangle$?

$$\langle\hat{N}^2\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2\langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle,$$

but $\hat{a}\hat{a}^\dagger = [\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}$, so, since $[\hat{a},\hat{a}^\dagger] = 1$,

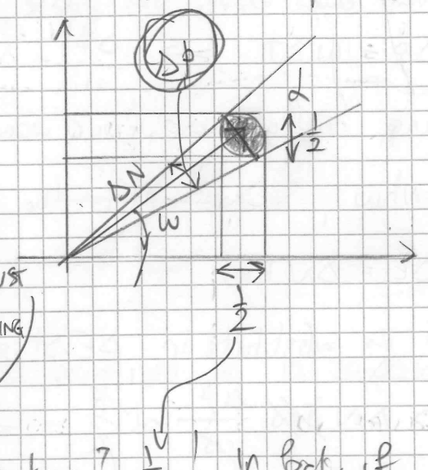
$$\Rightarrow = |\alpha|^2 \langle \alpha | 1 + \hat{a}^\dagger \hat{a} | \alpha \rangle = = |\alpha|^2 + |\alpha|^4$$

Now, we can compute δN ! In fact,

$$\sqrt{(\delta N)^2_{|\alpha\rangle}} = \langle N^2 \rangle_{|\alpha\rangle} - \left(\langle N \rangle_{|\alpha\rangle} \right)^2 = \sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4} = |\alpha| \; !$$

So,

$\delta N = |\alpha|$, which happens to be the square root of $\langle N \rangle$.
So, the length of the vector represents δN !!! And its square, $\langle N \rangle$!



Now, thanks to this picture,
we can understand several things:
the uncertainty in $\hat{X}$ is the same as
that of the ground state! The same for Δp:

$$\Delta X = \frac{d_0}{\sqrt{2}} \qquad \Delta p = \frac{\hbar}{d_0 \sqrt{2}}$$

BECAUSE WE JUST MOVE THE STATE WITHOUT RESHAPING IT !!!

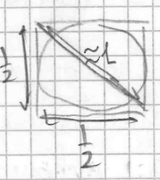
(their product should be $\frac{\hbar}{2}$;-)

How much is the uncertainty in this picture? $\frac{1}{2}$! In fact, if we roughly change $\langle X \rangle$ by $\frac{d_0}{\sqrt{2}}$, we have: $\frac{\Delta X}{\frac{d_0}{\sqrt{2}}} =$

$$= \frac{d_0}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}\, d_0} = \frac{1}{2}$$, For the vertical, if we change $\langle P \rangle$ by Δp,

$$\frac{\Delta p}{\sqrt{2}\,\hbar} d_0 = \frac{\hbar}{d_0 \sqrt{2}} \frac{d_0}{\sqrt{2}\,\hbar} = \frac{1}{2}$$

25 min: re-explained



(even wrong but fine)

Now: why we did this? Well, if we have this uncertainty, the phase is kinda ambiguous! We don't have a "well defined" phase of α: we have some uncertainty on it!
This is $\delta\phi$!
And, we can consider $\delta\phi$! Knowing that we have those $\frac{1}{2}$ lengths, saying that the blob radius is 1 (explained by me later)

$$\delta\phi \simeq \frac{1}{|\alpha|}$$ ⟶ radius of the blob; ⟶ length of the radius

But we know that $\delta N = |\alpha|$, so

$$\delta\phi = \frac{1}{\delta N} \Rightarrow \boxed{\Delta\phi \, \Delta N \approx 1}.$$

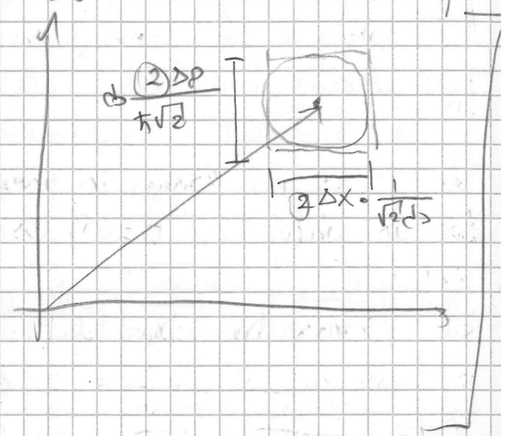
But this picture is more clear, as it provides some intuition of what $\delta\phi$ is and from where it originates. Not so accurate though.

What I would explain:

$$\langle \hat{X} \rangle \Rightarrow \langle \hat{X} \rangle \pm \Delta X$$

$$\langle \hat{P} \rangle \Rightarrow \langle \hat{P} \rangle \pm \Delta P,$$

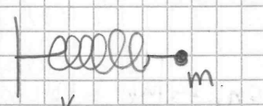
So the radius of the blob appears as 1, and everything goes well



This could be done more rigorously by the theory of Wigner distributions ;-)
Change of subject!

Squeezed states 20 min

Let us imagine to have a harmonic oscillator, a particle with mass m_1 and a spring,



, and our Hamiltonian

$$\hat{H}_1 = \frac{\hat{P}^2}{2m_1} + \frac{1}{2} m_1 \omega_1 \hat{X}^2, \qquad t < 0$$

and momentum

and we observe this Hamiltonian. Position ~~has~~ our uncertainty

$$\Delta x = \sqrt{\frac{\hbar}{2 m_1 \omega_1}} \qquad \Delta p = \sqrt{\frac{\hbar m_1 \omega_1}{2}},$$

and, if we are in the ground state, the bound is saturated:

$$\Delta x \, \Delta p = \frac{\hbar}{2}$$

Then, imagine that something in the physics changes drastically such as an abrupt temperature variation, affecting ω_1, K_1, m_1 ...
... the Hamiltonian $\hat{H}_1$, for $t < 0$, at $t = 0$ changes into $\hat{H}_2$. Then, we have

$$\hat{H}_2 = \frac{\hat{p}^2}{2m_2} + \frac{1}{2} m_2 \omega_2^2 \hat{x}^2 \qquad t = 0$$

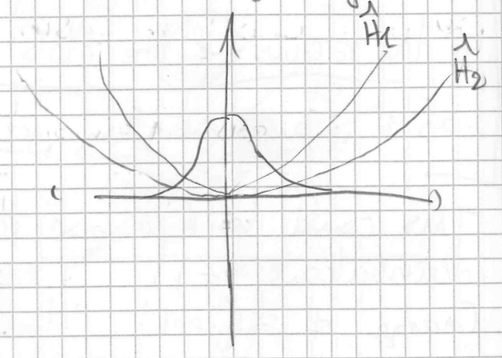
So: for $t < 0$, the particle is in the ground state. Then, at $t = 0$, the physics of the system change suddenly.
This means that, at $t = 0$ the wavefunction of $t < 0$ finds itself in a different Hamiltonian!
So, it is no longer in the ground state!
Let us write for $t = 0^+$, Δx and Δp in the following way:

$$\Delta x = \sqrt{\frac{\hbar}{2 m_2 \omega_2}} \sqrt{\frac{m_2 \omega_2}{m_1 \omega_1}} = e^{-\gamma} \sqrt{\frac{\hbar}{2 m_2 \omega_2}}$$

$$\Delta p = \sqrt{\frac{\hbar m_2 \omega_2}{2}} \sqrt{\frac{m_1 \omega_1}{m_2 \omega_2}} = e^{+\gamma} \sqrt{\frac{\hbar m_2 \omega_2}{2}},$$

where we defined this $e^{\pm\gamma}$!

$$e^{\gamma} = \sqrt{\frac{m_1 \omega_1}{m_2 \omega_2}} !$$



From the point of view of the second Hamiltonian, the former uncertainties were not right: they changed! So, what happened, is that the uncertainty gets squeezed by the change of Hamiltonian!
If Δx is squeezed, Δp is consequently stretched!

So, from the point of view of the second Hamiltonian, the ground state of the first Hamiltonian is a squeezed state: a state whose uncertainty has been squeezed.
Let us calculate and understand these things, and how evolution occurs.

The $\hat{x}$ and $\hat{p}$ operators are what describe the particle. In this sense, they are unchanged. So, by recalling our old formulas,

$$\hat{x} = \sqrt{\frac{\hbar}{2 m_1 \omega_1}} \left(\hat{a}_1 + \hat{a}_1^\dagger \right) = \sqrt{\frac{\hbar}{2 m_2 \omega_2}} \left(\hat{a}_2 + \hat{a}_2^\dagger \right)$$

$$\hat{p} = -i \sqrt{\frac{m_1 \omega_1 \hbar}{2}} \left(\hat{a}_1 - \hat{a}_1^\dagger \right) = -i \sqrt{\frac{m_2 \omega_2 \hbar}{2}} \left(\hat{a}_2 - \hat{a}_2^\dagger \right)$$

VERY STRONG ASSUMPTIONS: we can build $\hat{x}$ by both ingredients of the first and the second Hamiltonians!

So, we are not talking about two particles with their own $\hat{x}_1, \hat{p}_1$ and $\hat{x}_2, \hat{p}_2$: the SAME particle, with different Hamiltonians!
These conditions can be developed, leading to

$$\hat{a}_1 + \hat{a}_1^\dagger = e^{\gamma} \left(\hat{a}_2 + \hat{a}_2^\dagger \right)$$

$$\hat{a}_1 - \hat{a}_1^\dagger = e^{-\gamma} \left(\hat{a}_2 - \hat{a}_2^\dagger \right),$$

and we can solve, leading to

$$\hat{a}_1 = \hat{a}_2 \cosh\gamma + \hat{a}_2^\dagger \sinh\gamma \qquad (a)$$

$$\hat{a}_1^\dagger = \hat{a}_2^\dagger \cosh\gamma + \hat{a}_2 \sinh\gamma \qquad (b) \qquad (1)$$

This means that the annihilation operator for the first Hamiltonian is a mixture of creation and annihilation of the 2nd!
This relation is commonly referred to as Bogoliubov transformation!
It is quite simple to find the inverse transform: $\gamma \to -\gamma$ (the exponential goes in the LHS).

Now, we want to calculate, from the ground state of the 1st oscillator, $|0\rangle_{(1)}$, how it is written as a superposition of states of the 2nd oscillator!

$$|0\rangle_{(1)} = c_0 |0\rangle_{(2)} + c_2 \hat{a}_2^\dagger \hat{a}_2^\dagger |0\rangle_{(2)} + c_4 \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2^\dagger |0\rangle_2 + \dots \qquad (2)$$

Since the wavefunction of $|0\rangle_{(1)}$ is even, it should be written as a superposition of even states, so two creations per time: even occupation numbers!
What we know about $|0\rangle_{(1)}$ is that

$$\hat{a}_1 |0\rangle_{(1)} = 0,$$

but from eq. (1),

$$\hat{a}_1 = \hat{a}_2 \cosh\gamma + \hat{a}_2^\dagger \sinh\gamma,$$

so,

$$\left(\hat{a}_2 \cosh\gamma + \hat{a}_2^\dagger \sinh\gamma\right)|\phi\rangle_{(1)} = 0 \qquad (3)$$

We should put up terms in the RHS $\neq 0$) and somehow figure out what kills $|\phi\rangle_{(1)}$. But, we have something, which will help us. Let us set an ansatz on this $|\phi\rangle_{(1)}$:

$$|\phi\rangle_{(1)} = n(\gamma)\left\{\qquad\qquad\right\}|\phi\rangle_{(2)}$$

a normalization constant, $\{$ something really messy $\}$ and $|\phi\rangle_{(2)}$

What is lying in the curly brackets? Well, we learned of coherent states that are "exponentials of oscillators", exponentials of $\hat{a}$ and $\hat{a}^\dagger$, so let us try to attempt to put something like

$$|\phi\rangle_{(1)} = n(\gamma)\exp\left(-\frac{1}{2}f(\gamma)\hat{a}_2^\dagger \hat{a}_2^\dagger\right)|\phi\rangle_{(2)} \qquad (4)$$

($f(\gamma)$: a coefficient; $\hat{a}_2^\dagger\hat{a}_2^\dagger$: 2 creations to have something even)

Can we solve this? meaning: is this ansatz (4) annihilated by our operator (3)? We have to see if

$$n(\gamma)\left(\hat{a}_2\cosh\gamma + \hat{a}_2^\dagger\sinh\gamma\right)\exp\left(-\frac{1}{2}f(\gamma)\hat{a}_2^\dagger\hat{a}_2^\dagger\right)|\phi\rangle_{(2)} = 0$$

Let us see: this $\hat{a}_2$ commutes with the exponential (as it has $\hat{a}_2^\dagger$ as argument), so we can bring it on the right of the exponential, and...

Nothing! $\hat{a}_2^\dagger$ does not kill $|\phi\rangle_{(2)} = 0$

Let us start from the annihilating part, with $\hat{a}_2$!

Universal fact: if we have $\hat{a}$ and $|\phi\rangle$, it is ALWAYS true that

$$\hat{a}\hat{\mathcal{O}}|\phi\rangle = [\hat{a},\mathcal{O}]|\phi\rangle + \hat{\mathcal{O}}\hat{a}|\phi\rangle$$

because $[\hat{a},\mathcal{O}] = \hat{a}\hat{\mathcal{O}} - \hat{\mathcal{O}}\hat{a}$, which kills the vacuum

So,

$$\cosh\gamma\left[\hat{a}_2, \exp\left(-\frac{1}{2}f(\gamma)\hat{a}_2^\dagger\hat{a}_2^\dagger\right)\right]|\phi\rangle_{(2)} + \sinh\gamma\,\hat{a}_2^\dagger\exp\left(-\frac{1}{2}f(\gamma)\hat{a}_2^\dagger\hat{a}_2^\dagger\right)|\phi\rangle_{(2)} = 0$$

The commutator is quite simple; in fact, since

$$[\hat{A}, e^{\hat{B}}] = [A,B]e^{\hat{B}} \quad \text{if} \quad [[A,B],B] = 0,$$

and this is the case!

In fact,

$$\left[\hat{a}, \hat{a}^\dagger\hat{a}^\dagger\right] = c\,\hat{a}^\dagger, \quad \text{and } \hat{a} \text{ commutes with } \hat{a}^\dagger\hat{a}^\dagger \text{ (!-)}$$

So, we have

$$\Rightarrow \cosh\gamma\left[\hat{a}_2, -\frac{1}{2}f(\gamma)\hat{a}_2^\dagger\hat{a}_2^\dagger\right]\exp\left(-\frac{1}{2}f(\gamma)\hat{a}_2^\dagger\hat{a}_2^\dagger\right)|\phi\rangle_{(2)} +$$

$$+\sinh(\gamma)\,\hat{a}_2^\dagger\exp(\dots)|\phi\rangle_{(2)} = 0$$

Then,

$$\left[\hat{a}_2, -\frac{1}{2}f(\gamma)\hat{a}_2^\dagger\hat{a}_2^\dagger\right] = -\cosh\gamma\, f(\gamma)\hat{a}_2^\dagger$$

So, we have

$$\left(-\cosh\gamma\, f(\gamma)\hat{a}_2^\dagger + \sinh\gamma\,\hat{a}_2^\dagger\right)\exp(\dots)|\phi\rangle_{(2)} = 0$$

These two terms cancel if

$$f(\gamma) = \tanh(\gamma) \qquad !!! \qquad \left(\tanh(\gamma) = \frac{\sinh(\gamma)}{\cosh(\gamma)}\right)$$

So, our states end up to be

$$|\phi\rangle_{(1)} = n(\gamma)\exp\left(-\frac{1}{2}\tanh\gamma\,\hat{a}_2^\dagger\hat{a}_2^\dagger\right)|\phi\rangle_{(2)},$$

which is a transformation of the state from a basis of a Hilbert space, $|\phi\rangle_{(2)}$, to that of another Hilbert space, $|\phi\rangle_{(1)}$!!

It is not straightforward to get $n(\gamma)$. The most naive way is saying that it has unit norm, so

$$n^2(\gamma)\,\langle\phi|\exp\left(-\frac{1}{2}\tanh\gamma\,\hat{a}_2\hat{a}_2\right)\exp\left(-\frac{1}{2}\tanh\gamma\,\hat{a}_2^\dagger\hat{a}_2^\dagger\right)|\phi\rangle = 1$$

But this computation is very difficult.

Another surprising way to proceed is to compute

$$\underset{(2)}{\langle 0|0\rangle}_{(1)} = n(\gamma)\,{}_{(2)}\langle 0|e^{-\frac{1}{2}\tanh\gamma\,\hat{a}_2^{\dagger}\hat{a}_2^{\dagger}}|0\rangle$$

This equals 1. In fact, if we expand the exponential, we have some ${}_{(2)}\langle 0|\hat{a}_2^{\dagger}$ (, what equal 0! Just the 1 remains!

So,

$$n(\gamma) = \underset{(2)}{\langle 0|0\rangle}_{(1)}$$

If we introduce a complete set of position states, this is equal to

$$= \int dx \;\underset{(2)}{\langle 0|x\rangle\langle x|0\rangle}_{(1)} = \int dx \left(\psi_0^{(2)*}(x)\,\psi_0^{(1)}(x)\right),$$

and WE KNOW the wavefunctions of the two Hamiltonians!
So, this equals:

$$= \frac{1}{\sqrt{\cosh(\gamma)}}$$

Finally our squeezed states are:

$$|0\rangle_{(1)} = \frac{1}{\sqrt{\cosh\gamma}}\exp\left(-\frac{1}{2}\tanh\gamma\,\hat{a}_2^{\dagger}\hat{a}_2^{\dagger}\right)|0\rangle_{(2)} \;(,$$

where γ is the "squeezing parameter"!

Let us start from the beginning. Let us assume to have a Hamiltonian $\hat{H}$, its vacuum state $|0\rangle$ and m, ω. We define its "squeezed vacuum" $|0_\gamma\rangle$ to be:

$$|0_\gamma\rangle = \frac{1}{\sqrt{\cosh\gamma}}\exp\left(-\frac{1}{2}\tanh\gamma\,\hat{a}^{\dagger}\hat{a}^{\dagger}\right)|0\rangle, \quad \text{with}$$

$$\Delta x = e^{-\gamma}\sqrt{\frac{\hbar}{2m\omega}} \qquad \Delta p = e^{\gamma}\sqrt{\frac{m\hbar\omega}{2}}, \quad \text{for a given } \gamma.$$

The previous explanation was good to understand what squeezed states are and how this γ originates, but it is not required to distinguish these two Hamiltonians!

It is to be remarked that $|0_\gamma\rangle$ must come from the action of some unitary operator acting on the vacuum, since $\langle 0_\gamma|0_\gamma\rangle = 1$! as well as $\langle 0|0\rangle = 1$!

This unitary operator is not so easy to find; it is:

$$S(\gamma) = \frac{1}{\sqrt{\cosh\gamma}}\exp\left(-\frac{1}{2}\tanh\gamma\,\hat{a}^{\dagger}\hat{a}^{\dagger}\right)$$

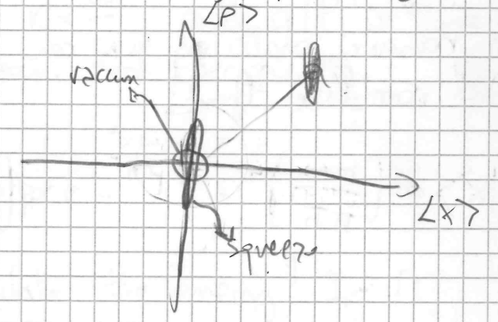
this operator should be written as something like $e^{\left(\hat{a}^{\dagger}\hat{a}^{\dagger} - \hat{a}\hat{a}\right)}$ to be unitary :-. To achieve such an equation, we could assume small γ, or doing something similar, to find

$$\hat{S}(\gamma) = \exp\left(-\frac{\gamma}{2}\left(\hat{a}^{\dagger}\hat{a}^{\dagger} - \hat{a}\hat{a}\right)\right)\frac{1}{\sqrt{\cosh\gamma}} \qquad \text{Squeezing operator}$$

$$|0_\gamma\rangle = S(\gamma)|0\rangle$$

The generalization of these states is obtained by squeezing and then translating these states:

$$\hat{D}(\alpha)\hat{S}(\gamma)|0\rangle = |\alpha,\gamma\rangle$$



Squeezing: reduces Δx
Translating: translates it!

By this way we can stabilize the state in position!

Squeezed states can be used to generate strange states that are now going to be described.
Let us squeeze a state "to the end": $\gamma \to \infty$!!
The state tends to be singular, as $\cosh\gamma \to 0$. So:

$$|0_\infty\rangle \sim \exp\left(-\frac{1}{2}\hat{a}^{\dagger}\hat{a}^{\dagger}\right)|0\rangle$$

these states are reasonable because $\tanh \in [-1,1]$

and thus tends to $\delta(x)$ $x=0$
In fact, this is like a position state!

Since

$$\hat{X} = \hat{a} + \hat{a}^\dagger, \qquad \left(\hat{a} + \hat{a}^\dagger\right) \exp\left(-\frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger\right)|0\rangle = \left(\hat{a} - \hat{a}\right) \to 0$$

(Computing the commutator)

$|0\rangle$ is the eigenvalue of $\hat{X}$ on $\delta(x)$ (-)

Instead,

$|0_{-\infty}\rangle$ is $\exp\left(+\frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger\right)$, which is a $\delta(p)$: δ in MOMENTUM!

uncertainty of momentum $\to 0$ ($\sim e^{\gamma}$, $\gamma \to -\infty$)

This allows to build position and momentum operators BY SQUEEZING STATES!

So,

$$|x\rangle = \mathcal{N} \exp\left(\sqrt{\frac{2m\omega}{\hbar}} x \hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger\right)|0\rangle$$

to check this, we can see that

$$\hat{X}\left(\{\}\right)\left(\hat{a} + \hat{a}^\dagger\right) \qquad \hat{X}|x\rangle = x|x\rangle$$

In Harmonic oscillators we can then build position eigenstates! And normalize them:

$$|x\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} \exp\left(\sqrt{\frac{2m\omega}{\hbar}} x \hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger\right)|0\rangle$$

↑ displacement

Lecture 16: Quantum Dynamics and Two State Systems

Photon states

Photon states are quantum states of the electromagnetic field: quantum field theory! A first introduction, at least!

A key identity in this context is the definition of coherent states as displacement of the vacuum,

$$|\alpha\rangle = D(\alpha)|0\rangle, \qquad D(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}},$$

and

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

Those were coherent states. Now, PHOTON states!

Photon states have to do with the electromagnetic field. In particular, a very important quantity when discussing electromagnetic fields is their energy! And energy have to do with Hamiltonians! So, this could be a starting point, to introduce a quantum description of the EM field. In an EM field, energy is something like (Poynting vector)

$$E = \frac{1}{2}\int_V dx^3\, \varepsilon_0\left[\vec{E}^2(\vec{r},t) + c^2\vec{B}^2(\vec{r},t)\right] \qquad (1)$$

Let us try to describe a particular configuration of EM fields: a single mode. This is like having some sort of cavity, with a finite volume, and a "wave" in it. This could have the form

$$E_x(z,t) = \sqrt{\frac{2}{\varepsilon_0 V}}\, \omega q(t) \sin(kz) \qquad (2a)$$

where V is the volume of the "cavity", $k = \frac{\omega}{c}$, and $q(t)$ is some function of time, to be determined (I mean, "not specified").

There is also a magnetic component:

$$cB_y(z,t) = \sqrt{\frac{2}{\varepsilon_0 V}}\, p(t)\cos(kz) \qquad (2b)$$

Given these (2), we can try to calculate the energy! In (1), we have to compute this volume integral, which might depend on time. By recalling that

$$\int_V \sin^2(kz)\, dx^3 = \frac{1}{2},$$

we find that

$$E = \frac{1}{2}\left(p^2(t) + \omega^2 q^2(t)\right).$$

This is very, very interesting! At first, such a formula should remind something to us: this looks like a harmonic oscillator! Except that the mass is gone!

Units seem to be wrong: we know that $\frac{p^2}{2m}$ has units of an energy, but this doesn't!
This is actually very good, because photons have NO MASS! So, it is nice NOT to have a mass in this formula!

On the other hand, we can say, by looking (2), that

$[p] = \sqrt{\text{energy}}$ (square root of energy)

$[q] = \text{time}\sqrt{\text{energy}}$

So, "p" and "q" are not exactly what $\hat{p}$ and $\hat{q}$ were, BUT,

$[pq] = \text{time} \times \text{energy}$, which are the units of $\hbar$!!!

All of this should inspire us: we have currently no idea of what a "quantum of an EM field is", but we found a natural correspondence between one EM mode and an energy functional that looks very similar to an harmonic oscillator. So, we can DECLARE these things to be a Hamiltonian and these $p(t)$, $q(t)$ to be the Heisenberg operators of the EM field. So, let us DEFINE the Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2\right) \qquad (3)$$

Now: these steps could seem quite "speculative", but they are actually reasonable. We can check it: we can recall that the quantum equations of motion, of Heisenberg operators, should look like classical equations of motion. So, let us compute the Heisenberg equations of motion for these operators:

$$i\hbar \frac{d}{dt}\hat{p}_H(t) = \left[\hat{H}, \hat{p}_H\right] \qquad \text{(more or less)}$$

But, we also know Maxwell's equations, and plug (2) in them and check (HOMEWORK). Let us now "accept" this °_°.
Then, accepted also that $p(t)$ and $q(t)$ have funny units, the claim is that we can take all our previous formulas and write them with $\boxed{m=1}$, to re-use them here! In particular, we can write:

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{p} = \sqrt{\frac{\hbar\omega}{2}}\frac{1}{i}\left(\hat{a} - \hat{a}^\dagger\right)$$

Just like $\hat{x}$ and $\hat{p}$, without names!

We can now write (3) in terms of the number operator:

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = !$$

$$= \hbar\omega\left(\hat{N} + \frac{1}{2}\right).$$

This allows us to obtain the physical interpretation that if we have states with some number operator, the energy is exactly that number, times $\hbar\omega$!!! And this is why we think about photons! If we have N photons in a given state, we would have an energy $N\hbar\omega$.
This is a dramatic assumption, something very hard: quantum field theory! Somehow, $q(t)$ and $p(t)$, which have ABSOLUTELY NOTHING TO DO with position and momentum as we always discussed them. Actually, q is something related to the electric field E, and p is somehow close to B, so it is just like E and B have become quantum operators. Quantum field theory is the idea that the fields become operators. And it seems to be right in the sense that the state with N photons can be viewed as a state of the harmonic oscillator, just like the usual one, with mass equal to 1.
So, we can turn this formula to its Heisenberg form without really doing any calculation, but just recalling old results, so that we can write

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}\left[e^{-i\omega t}\hat{a} + e^{+i\omega t}\hat{a}^\dagger\right]$$

and, given that, we can back-substitute in our electric field (2), so to achieve

$$\hat{E}_x(z,t) = \mathcal{E}_0\left(e^{-i\omega t}\hat{a} + e^{+i\omega t}\hat{a}^\dagger\right)\sin(kz), \quad \text{with } \mathcal{E}_0 = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}.$$

This is the electromagnetic field operator.
If we want to get some intuition about this field operator, let us try to find its expectation value. Let $|n\rangle$ be an eigenstate of the harmonic oscillator with occupation number "n", so a state of n photons. This is also an energy eigenstate, with

energy $E_n = n\hbar\omega + \frac{1}{2}\hbar\omega$.
Then,

$$\langle \hat{E}_x \rangle_{|n\rangle} = \mathcal{E}_0 \left(e^{-i\omega t} \langle n|\hat{a}|n\rangle + e^{+i\omega t} \langle n|\hat{a}^\dagger|n\rangle \right) \sin(kz).$$

In other words: we have an EM field, we put it in a state, and then we ask: how does the field look in that state?
Well,

$$\langle n|\hat{a}|n\rangle = 0. \text{ In fact,}$$

$$\hat{a}|n\rangle = |n-1\rangle,$$

and

$$\langle n|n-1\rangle = 0, \text{ for they are orthogonal.}$$

The same for

$$\langle n|\hat{a}^\dagger|n\rangle = \langle n|n+1\rangle = 0.$$

So, $\langle \hat{E}_x \rangle_{|n\rangle} = 0$
This is actually not too strange; in fact, energy eigenstates are very unintuitive objects: nothing very interesting happens.
Let us take now a more thoughtful state: let us put a coherent state of photons into this:

$|\alpha\rangle$ (coherent state),

$$\hookrightarrow \langle \hat{E}_x \rangle_{|\alpha\rangle} = \mathcal{E}_0 \left(e^{-i\omega t} \langle \alpha|\hat{a}|\alpha\rangle + e^{+i\omega t} \langle \alpha|\hat{a}^\dagger|\alpha\rangle \right) \sin(kz).$$

Now,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \text{ and}$$

$$\langle \alpha|\hat{a}|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha.$$

Similarly, $\langle \alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^*$. So,

$$\langle \hat{E}_x \rangle_{|\alpha\rangle} = \mathcal{E}_0 \left(\alpha e^{-i\omega t} + \alpha^* e^{+i\omega t} \right) \sin kz.$$

This is telling us that the coherent state is the state for which the expectation value of the electromagnetic field is precisely the kind of EM waves that we know so well. This, just because only on coherent states $\hat{a}$ and $\hat{a}^\dagger$ have expectation value.
So, what we call a "classical wave" resonating in a cavity, is a coherent state of the electromagnetic field in this sense. They of course have uncertainties, but they have a nice, classical picture!
Lasers for instance are coherent states of the EM field. With all the uncertainties on N and ϕ that we discussed last time.
We can still develop our formulas, and obtain:

$$= 2\mathcal{E}_0 |\alpha| \cos\{\alpha e^{-i\omega t}\} \sin kz,$$

and if we write $\alpha = |\alpha| e^{i\theta}$, then

$$\hookrightarrow = 2\mathcal{E}_0 |\alpha| \cos(\omega t - \theta) \sin(kz),$$

and this is something like a standing wave, with a constant phase distribution!
Then, the expectation value of the Hamiltonian is:

$$E = \langle H \rangle_{|\alpha\rangle} = \hbar\omega \left(\langle \hat{N} \rangle + \frac{1}{2} \right) = \hbar\omega \left(|\alpha|^2 + \frac{1}{2} \right)$$

↳ last time!

So, a coherent state $|\alpha\rangle$ has $N = |\alpha|^2$ photons!
This was just with 1 mode. We could do this with other modes and put everything together!
So different energy levels correspond to different amounts of photons in them!
The energy of every mode correspond to the number of photons for that mode (I guess)

Two-state systems

First, we want to understand spin precession. Spins, magnetic fields...
This seems to be a very specific problem. BUT, ACTUALLY, any two-state system can be seen as a system where we put a spin in magnetic field.
For instance, electrons shared between 2 atoms, or ammonia molecule.
The math is the SAME!

Spin precession

The idea of spin precession arises from a charged particle, which has spin, meaning that there's a relation between its magnetic moment $\vec{\mu}$ and its spin:

$$\vec{\mu} = \frac{q}{2m}\vec{S} \qquad \vec{S}: \text{angular momentum}$$

This idea in quantum mechanics is roughly right, except for two modifications:

$$\vec{\mu} = g\frac{q}{2m}\vec{S}$$

: there's a "g" factor, and $\vec{S}$ is not a plain classical angular momentum, a rotating ball with some radius, but it's SPIN angular momentum

$\vec{S}$ should be best viewed as an operator, so that $\vec{\mu}$ becomes an operator as well, and, depending on the particle, g assumes different values! For the electron, $g=2$, as also predicted by Dirac's equation, while for protons or neutrons it changes!

At the end of the day, we want to write

$\vec{\mu} = \gamma\vec{S}$, where γ contains q, g, m, and so on!

So, the Hamiltonian

$\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B}$, where $\hat{\vec{\mu}}$ is an operator, and for now $\vec{B}$ is a static magnetic field

$$= -\gamma\hat{\vec{B}}\cdot\hat{\vec{S}} =$$

$$= -\gamma\left(B_x\hat{S}_x + B_y\hat{S}_y + B_z\hat{S}_z\right) \quad !$$

Let us recall the case of a magnetic field oriented along $\hat{z}$ direction. Then, if $\vec{B} = B_z\hat{z}$,

$\hat{H} = -\gamma B\hat{S}_z$!

And the unitary operator that generates the time evolution of states, $U(t,0)$, is:

$$U(t,0) = \exp\left(-i\frac{H_S t}{\hbar}\right) = \exp\left(-i\frac{(-\gamma B + S_z)t}{\hbar}\right) \qquad (4)$$

Let us now recall a property [from HOMEWORKS]: we talked about one operator

$$\hat{R}_{\vec{n}}(\alpha) = \exp\left(-\frac{i\alpha\hat{S}_{\vec{n}}}{\hbar}\right), \quad \hat{S}_{\vec{n}} \text{ is } \hat{n}\cdot\hat{\vec{S}} = n_x\hat{S}_x + n_y\hat{S}_y + n_z\hat{S}_z, \quad \hat{n} \text{ a unit vector}$$

This operator was called "rotation operator", and it did perform rotation of spin states! It rotated any spin state of an angle α around the $\hat{n}$ direction!

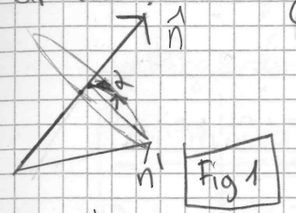
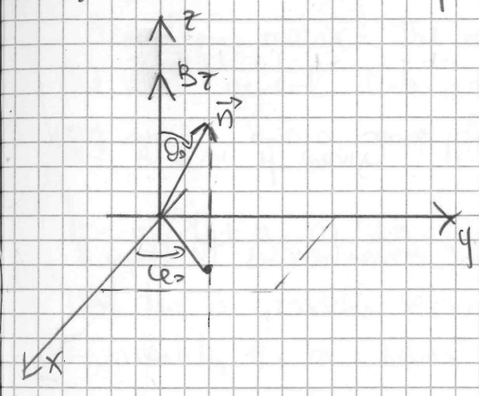


Fig 1

Given a spin state $|n', \pm\rangle$, for instance $|n', +\rangle$, $\hat{R}_{\vec{n}}(\alpha)$ rotates $|n', +\rangle$ by α, using an axis n

Now! the operator in (4) is rather similar to $R_n(\alpha)$! Where $\alpha = -\gamma B t$, S_z is $S_{\vec{n}}$! Let us prove this! For an arbitrary spin state! where $\vec{B} = B_z\hat{z}$, and our spin state is oriented along $\vec{n}$!



Then,

$$|\Psi, 0\rangle = \cos\frac{\theta_0}{2}|+\rangle + \sin\frac{\theta_0}{2}e^{i\phi_0}|-\rangle.$$

(from formulas ... one of the first lectures!

Before applying our time-evolution operator, let's do a preliminary calculation

$$H_S|+\rangle = -\gamma B S_z|+\rangle = -\frac{\gamma B\hbar}{2}|+\rangle$$

$$H_S|-\rangle = -\gamma B S_z|-\rangle = +\gamma B\frac{\hbar}{2}|-\rangle$$

So,

$$|\Psi, t\rangle = \exp\left(-i\frac{H_S t}{\hbar}\right)|\Psi, 0\rangle = \cos\frac{\theta_0}{2}\exp\left(-i\frac{H_S t}{\hbar}\right)|+\rangle + \sin\frac{\theta_0}{2}\cdots$$

But, with the Taylor series and so on, it becomes:

$$= \cos\frac{\theta_0}{2}\exp\left(-i(\gamma B\frac{\hbar}{2}\frac{1}{\hbar}t)\right)|+\rangle + \sin\frac{\theta_0}{2}e^{i\phi_0}\exp\left(i\gamma B\frac{\hbar}{2}\frac{1}{\hbar}t\right)|-\rangle$$

So,

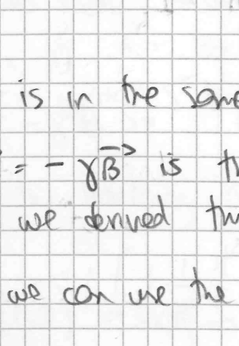
$$|\Psi, t\rangle = \cos\frac{\theta_0}{2}\exp\left(+i\gamma B\frac{t}{2}\right)|+\rangle + \sin\frac{\theta_0}{2}e^{i\phi_0}\exp\left(-i\gamma B\frac{t}{2}\right)$$

These states are not so readable, so we could factor out some phase shift:

$$= \exp\left(i\gamma B \frac{t}{2}\right)\left(\cos\frac{\vartheta_0}{2}|+\rangle + \sin\frac{\vartheta_0}{2}\exp(\phi_0 - \gamma B t)|-\rangle\right)$$

This is a state where $\vartheta(t) = \vartheta_0$, and $\phi(t) = \phi_0 - \gamma B t$!

This spin will precess and rotate as in Fig. 1 // Fig 2 below



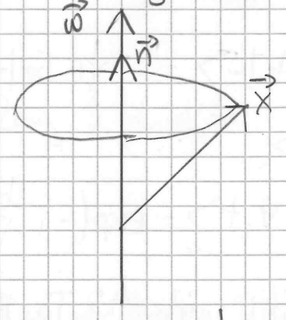
$\phi(t) = \phi_0 - \gamma B t$! Rotating around the axis defined by $\vec{B} = B_z \vec{z}$!

So: classical picture of spin: could we use it? Given $\vec{\mu}$ in a $\vec{B}$ field, we get a torque, given by $\vec{\mu} \times \vec{B} = \vec{\tau}$
On the other hand, the rate of change of angular momentum is the torque $\vec{\tau}$,

$$\frac{d\vec{S}}{dt} = \vec{\tau} = \vec{\mu} \times \vec{B} = -(\gamma \vec{B}) \times \vec{S} \quad (5)$$

↳ this equation is a very particular case of a famous equation in classical mechanics, which is the equation for a rotating vector!
Given a vector $\vec{x}$,

$$\frac{d\vec{x}}{dt} = \vec{\omega} \times \vec{x} \quad ! \qquad \vec{\omega}: \text{ angular frequency}$$



(5) is in the same form! Here,
$\vec{\omega} = -\gamma \vec{B}$ is the "Larmor frequency"!
And we derived this by both classical and quantum mechanics!

Now we can use the Larmor frequency to rewrite the Hamiltonian, so:

$$H_S = -\vec{\mu} \cdot \vec{B} = -\gamma \vec{B} \cdot \hat{\vec{S}} = \vec{\omega_L} \cdot \hat{\vec{S}}$$

and if someone gives us a Hamiltonian that we can write as "some vector dot $\vec{S}$", that some vector is, thinking of spins, the Larmor frequency of rotation.

This was a general discussion of precession in a magnetic field. Let us generalize it more: any system with 2 states can be seen as a spin system.

The most general Hamiltonian for a two-state system is:

$$\hat{H} = \begin{pmatrix} g_0 + g_3 & g_1 - ig_2 \\ g_1 + ig_2 & g_0 - g_3 \end{pmatrix}$$

two-state system: a system with TWO BASIS STATES, which, combined, of course can produce ∞^2 states!

↓

Hermitian matrix! The most general form of a Hermitian matrix! The most general time-independent 2x2 Hamiltonian! However, we can write $\hat{H}$ in a more friendly way: this can be written as

$$\hat{H} = g_0 \mathbb{1} + g_1 \underline{\underline{\sigma}}_1 + g_2 \underline{\underline{\sigma}}_2 + g_3 \underline{\underline{\sigma}}_3,$$

and this is possible because the Pauli matrices $\underline{\underline{\sigma}}_i$ are, together with the identity $\mathbb{1}$, a basis for all Hermitian 2x2 matrices!

Now: we can further re-write it as

$$\hat{H} = g_0 \mathbb{1} + \vec{g} \cdot \underline{\underline{\vec{\sigma}}}, \qquad \vec{g} = (g_1\ g_2\ g_3) \qquad \underline{\underline{\vec{\sigma}}} = (\underline{\underline{\sigma}}_1\ \underline{\underline{\sigma}}_2\ \underline{\underline{\sigma}}_3)$$

we can rewrite

$$\vec{g} = g\vec{n}, \text{ such that } g = \sqrt{g_1^2 + g_2^2 + g_3^2} \qquad \vec{n} \cdot \vec{n} = 1,$$

$$\hat{H} = g_0 \mathbb{1} + g\vec{n} \cdot \underline{\underline{\vec{\sigma}}}$$

Now, we have to diagonalize $\hat{H}$, find eigenvalues, eigenvectors and so on, but this has already been done! In fact, the eigenstates of $\vec{n} \cdot \underline{\underline{\vec{\sigma}}}$ is:

$$\vec{n} \cdot \underline{\underline{\vec{\sigma}}} |n;\pm\rangle = \pm |n;\pm\rangle \qquad !!! \qquad (6)$$

In fact, if $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$,

$\vec{n}\cdot\vec{S}\,|n;\pm\rangle = \pm\frac{\hbar}{2}|n;\pm\rangle$,

but then $\vec{\sigma}$ matrices are already diagonalized by (6)!

And, the energy is:

$\hat{H}|n;\pm\rangle = (g_0\mathbb{1} + g\,\vec{n}\cdot\vec{\sigma})|n;\pm\rangle = (g_0 \pm g)|n;\pm\rangle$!

So, with $|n;+\rangle$, we have $E = g_0 + g$ as energy,

$|n;-\rangle$, we have $E = g_0 - g$

And we have thus $2g$ splitting among the two energies! $g_0 - g$ is the ground state, $g_0 + g$ is the excited state!

Last thing: we may want to study time evolution for the Hamiltonian

$\hat{H} = g_0\mathbb{1} + g\,\vec{n}\cdot\vec{\sigma}$

But we also have

$\hat{H}_S = \vec{\omega}_L\cdot\vec{S}$!

So,

$\vec{\omega}_L\cdot\vec{S}$ should be identified in $g_0\mathbb{1} + g\vec{n}\cdot\vec{\sigma}$!

Since

$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$,

$\hat{H} = g_0\mathbb{1} + \vec{g}\cdot\frac{2}{\hbar}\vec{S}$

$\vec{\omega}_L$

So in this system,

$\vec{\omega}_L = \frac{2}{\hbar}\vec{g}$

So $g_0\mathbb{1}$ is a constant term that produces a constant phase, whereas the other part "does precession"!

If we know the Hamiltonian of our system, then we know $\vec{g}$, and so we can treat this using this theory!! We know how it evolves in time!

Lecture 17: Two state systems

Now we will study a very important example of two-state system: the ammonia molecule. And this will teach us how MASERs work! Just like LASERs for light, @ microwaves!

Last time we spoke about the Hamiltonian of a 2-state system!

A 2×2 Hermitian matrix, with Pauli matrices...

$\hat{H} = g_0\mathbb{1} + \vec{g}\cdot\vec{\sigma} = g_0\mathbb{1} + \vec{\omega}_L\cdot\vec{S}$, ω_L is the Larmor frequency.

We explained that spins rotate with angular velocity ω_L, around the axis defined by the vector $\vec{\omega}_L$ with angular velocity $\|\vec{\omega}_L\|$: Larmor precession!

We also got that the energy levels for this Hamiltonian, which are related to two basis vectors, which are also energy eigenstates, $|\vec{n};+\rangle$ and $|\vec{n};-\rangle$, are $g_0 \pm g$!

We will call / define

$|+\rangle \equiv |1\rangle$ $\quad$ $|-\rangle \equiv |2\rangle$

However, even if physically the Hamiltonian has nothing to do with spins, we have the same notions valid, including the 2 basis vector $|1\rangle$ and $|2\rangle$ and we can perform the mathematical analogy with spin up ($|+\rangle$) and down ($|-\rangle$)

Ammonia Molecule

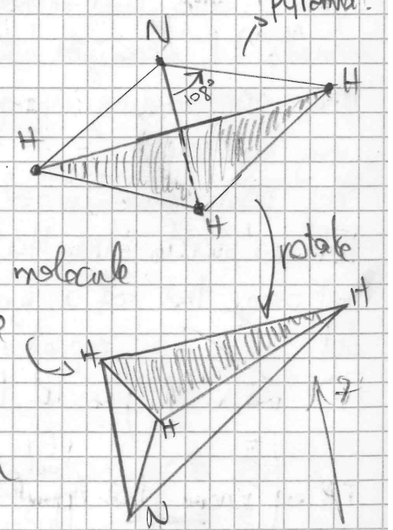
This is NH_3, used as a fertilizer, cleaning product...

Its shape is a flattened tetrahedron, where the angle is 108°.

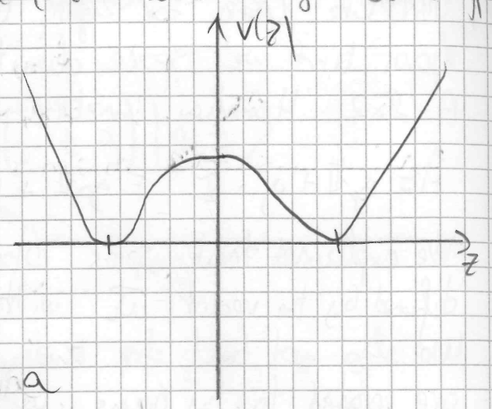
This is a 2-state system because just as the nitrogen can be above (as in this picture) the "basis" (the black-shadowed region) of the three hydrogens, it could be also BELOW it! The molecule can be imagined to be rotating, leading to the bottom figure.

So: pyramid where faces have 108° angle!

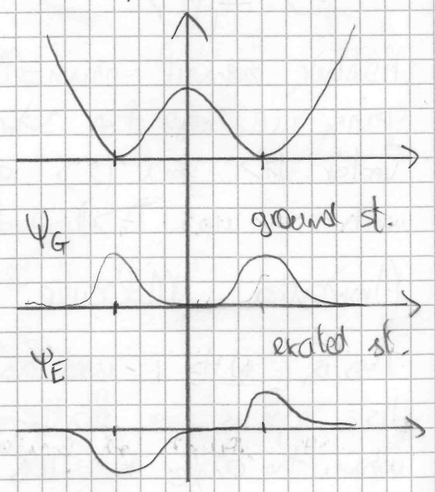
The configurations with nitrogen on top or on bottom are both possible. More specifically,



the top figure is a stable state but, if somehow we manage to push, to rotate it, well, it becomes stable in this new (bottom N) configuration. This is as if we had a potential for the nitrogen, $V(z)$, where z is the direction normal to the basis of the pyramid (-z means nitrogen down), where $V(z) = 0$ (stable state). For two configurations, but from one configuration to the other one finds a big barrier, which requires strong energy to be overcome.



Now, this is the qualitative description of ammonia. We will try to describe now it as a two-state system!
We need then some notation!
Let the first basis state $|1\rangle$ be called $|\uparrow\rangle$ (nitrogen up), and the second basis state, $|2\rangle$, be $|\downarrow\rangle$ (nitrogen down). Now, we can try to write the Hamiltonian for this system.
It's like having two quantum wells: the ground state has the two oscillations, same sign, and the excited state has oscillations with opposite sign (1 intersection in z axis: node theorem! ;-)



If the barrier is sufficiently high (in 2 QWs this means "distant wells") ("decoupled wells"), the two corresponding energy levels are not very different.
So: how do we model this?
Let us write the Hamiltonian. Let E_0 be a reference energy: for instance, that of a ground state. Surely, this will be in our Hamiltonian.
Let $|\uparrow\rangle$ be represented in matrix language with the vector $\begin{pmatrix}1\\0\end{pmatrix}$, and $|\downarrow\rangle$ with $\begin{pmatrix}0\\1\end{pmatrix}$. If the Hamiltonian was

$$H = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix},$$

then we would have E_0 as level of both $|\uparrow\rangle$ and $|\downarrow\rangle$, but this is not possible! we can not have degeneracies, so there MUST be something else happening.

Let us tinker on this Hamiltonian, to make the physics we are interested in appear. So, let us add constants $\Delta > 0$, as

$$\rightarrow H = \begin{bmatrix} E_0 & -\Delta \\ -\Delta & E_0 \end{bmatrix} \qquad (1) \qquad \Delta \in \mathbb{R}, \text{ for starter ;-)}$$

These Δ are the Hamiltonian matrix elements: $\langle 1|H|2\rangle$ and $\langle 2|H|1\rangle$. This $-\Delta$ cannot change the physics instead from $+\Delta$, because we could obtain $\begin{bmatrix} E_0 & \Delta \\ \Delta & E_0 \end{bmatrix}$ just by re-defining $|\uparrow\rangle \rightarrow -|\uparrow\rangle$, $|\downarrow\rangle \rightarrow -|\downarrow\rangle$: just a detail when choosing the basis states ;-)
Writing as (1), basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ are no longer energy eigenstates, for H is no longer diagonal!
It could be also interesting to figure out what this system has to do with the general representation. In other words, the Hamiltonian can be written as

$$H = \begin{bmatrix} E_0 & -\Delta \\ -\Delta & E_0 \end{bmatrix} = E_0 \mathbb{1} - \Delta \underline{\underline{\sigma_x}} \qquad !!!$$

So, comparing to the most general form of a two-state system, we have instead of $g\underline{\underline{\sigma}}$, just $g\underline{\underline{\sigma_x}}$ with magnitude Δ. And $\underline{\underline{\sigma_x}}$ is oriented along x" ;-)

$$\vec{g} = -\Delta \hat{x} \Rightarrow \boxed{g = \Delta}$$

So, the ground and excited states can be now computed, by using the formulas for the general case! Since we know that the energies are

$$E = g_0 \pm g, \quad \text{now we have}$$

$$E \begin{cases} E_0 + \Delta \rightarrow \text{Excited } (E_E) \\ E_0 - \Delta \rightarrow \text{Ground, } (E_G) \end{cases}$$

and the gap between these energies is then 2Δ.
It is not too hard to see that the eigenstate corresponding to E_E is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, while $E_G \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$. These are the eigenvectors!

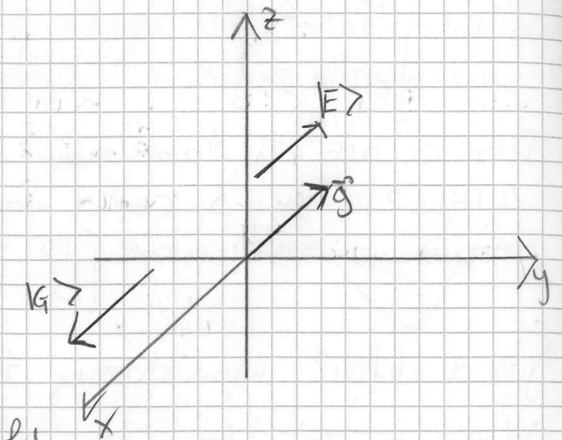
written in terms of our basis, we have:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

To interpret this in terms of spin states, $\vec{g}$ is on the x-axis, back-directed!

$$\vec{g} = -\Delta\hat{x}$$



With the general notation on spin, we identified $|\vec{n};+\rangle$ as the excited state, $|\vec{n};-\rangle$ as the ground. So, the excited state $|E\rangle$ is in the $-\hat{x}$ direction.
The ground state is in the $-\hat{n}$, so $-(-\hat{x})$ direction!

So, in spin language,

$$|E\rangle \propto |+\rangle - |-\rangle,$$

$$|G\rangle \propto |+\rangle + |-\rangle$$

The gap 2Δ is what we called $\hbar\omega$ for a photon: the "transition energy".
In particular, this 2Δ corresponds to a

$\nu = 23.87$ GHz, $\lambda \approx 1.26$ cm
This, for the ammonia molecule.
This "is known". Δ is a sort of perturbation splitting the excited and the ground states.

Now, exercise: how does the $|\uparrow\rangle$ state evolve in time?
The quickest way to answer to this question is to exploit the analogy with spins. In this case, the Larmor frequency is going to be around the direction of $\vec{g}$:

$$\vec{\omega}_L = \frac{2}{\hbar}\vec{g},$$

For this reason, by using the right-hand rule, $|\uparrow\rangle$ will precess with $\vec{\omega}_L$ vector parallel to $\vec{g}$, so to $\hat{x}$, so in the yz plane!

Let's do some calculations. If

$$|\Psi; 0\rangle = |\uparrow\rangle,$$

let us write $|\uparrow\rangle$ as a superposition of energy eigenstates $|G\rangle$ and $|E\rangle$:

$$|\uparrow\rangle \Rightarrow \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right)\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|E\rangle + |G\rangle\right).$$

Then, since we know the energies of these states ($E_0 \pm \Delta$), we know how they evolve in time with

$$e^{-\frac{iHt}{\hbar}},$$

and then re-write all the expressions in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$, as done in the lecture notes.

Final result,

$$|\Psi; t\rangle = e^{-\frac{iEt}{\hbar}}\left[\cos\left(\frac{\pm\Delta}{\hbar}\right)|\uparrow\rangle + i\sin\left(\frac{\pm\Delta}{\hbar}\right)|\downarrow\rangle\right]$$

For instance, from here, it is possible to compute the probability to be in $|\uparrow\rangle$ from the square of this:

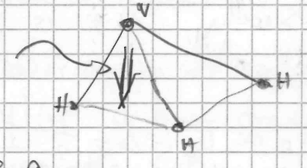
$$\cos^2\left(\frac{\pm\Delta}{\hbar}\right) \equiv P_{|\uparrow\rangle} \qquad \sin^2\left(\frac{\pm\Delta}{\hbar}\right) = P_{|\downarrow\rangle}$$

So, our pair $|\uparrow\rangle$, will start rotating like crazy, up and down, with 23.87 billion times /s speed!
This because $|\uparrow\rangle$ is not a stationary eigenstate! $|E\rangle$ or $|G\rangle$ are!
Notice that even if rotation occurs at $\frac{\Delta}{\hbar}$, Larmor frequency is

$$\omega_L = \frac{2\Delta}{\hbar}$$: the frequency of the photons!

No contradiction here! In fact, last lecture, we had $\cos\left(\frac{\theta}{2}\right)$, so fine!)

Let us try to make this more practical, and for this let us explore the fact that this molecule has an electric dipole moment! In fact, it's like: nitrogen on one side, and three hydrogens on the other!
So, there is an electric dipole moment pointing down: from the negative nitrogen region (which is negative because hydrogens' electrons kind of cluster around N).
So the "bottom" region is kind of positive. Then, we have a dipole moment pointing down!
Let $\vec{\mu}$ be this dipole moment! So, the energies can be written as the dipole moment, times the electric field:



$E = -\vec{\mu} \cdot \vec{\mathcal{E}}$ ⇝ electric field! Extend!

↳ energy contribution of our dipole placed in an electric field!

Let us now consider the case of an Hamiltonian with field in the z-direction (the same of the dipole moment), so $\vec{\mathcal{E}} = \mathcal{E}_z \hat{z}$

$\vec{\mu} = -\mu_z \hat{z}$,

↳ it points down, for $|\uparrow\rangle$ state

so, $\boxed{E = \mu_z \mathcal{E}_z}$ (double -),

and this is added to the other energies when the state is $|\uparrow\rangle$.

So, starting from our Hamiltonian

$\hat{H} = \begin{pmatrix} E_0 & -\Delta \\ -\Delta & E_0 \end{pmatrix}$, we should add

↳ $\hat{H} = \begin{bmatrix} E_0 + \mu\mathcal{E} & -\Delta \\ -\Delta & E_0 - \mu\mathcal{E} \end{bmatrix}$ → opposite behavior for $|\downarrow\rangle$! (2)

So, this is a reasonable expectation for the Hamiltonian of ammonia molecule inside an electric field!

First step: see that $\hat{H}$ can be written as

$\hat{H} = E_0 \mathbb{1} - \Delta \underline{\underline{\sigma}}_1 - \mu\mathcal{E}\, \underline{\underline{\sigma}}_3$

This is more complicated to be diagonalized, but, in the previous lecture, we solved the "general" problem! So, this time,

$E_E = E_0 + \|\vec{\Delta}\| = E_0 + \sqrt{\Delta^2 + (\mu\mathcal{E})^2}$

$E_G = E_0 - \sqrt{\Delta^2 + (\mu\mathcal{E})^2}$

NOTICE: $|E'\rangle$ and $|G'\rangle$ since they are different from $|E\rangle$ and $|G\rangle$! DEFORMED BY $\mathcal{E}$!

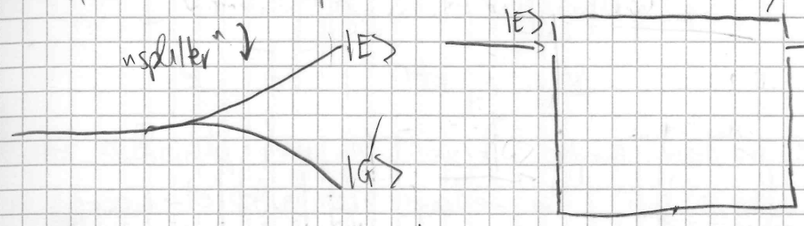
Usually, $\mu\mathcal{E}$ is much smaller than Δ (for standard electric fields).

So, we can approximate as

$E_E \approx E_0 + \Delta + \frac{1}{2}\frac{(\mu\mathcal{E})^2}{\Delta}$ $\quad \mu\mathcal{E} \ll \Delta$!

$E_G \approx E_0 - \Delta - \frac{1}{2}\frac{(\mu\mathcal{E})^2}{\Delta}$

This thing gives the opportunity (technologically) to separate a beam of particles into excited and ground-state levels. It's like a Stern-Gerlach machine, but not with spin! With "states"!



WE NEED AS INPUT ONLY EXCITED STATES (ammonia molecules)

Then, what we can do is to insert those two states in a resonant cavity, tuned to 23870 GHz (the frequency of the gap), and hope, starting from $|E\rangle$, to obtain some $|G\rangle$ in output!

The state $|E\rangle$ in input has energy $E_E = E_0 + \Delta$, since no electric field is assumed at the cavity input. If somehow we manage to grab $|G\rangle$ states out of the cavity, then they have $E_G = E_0 - \Delta$! And so they must have lost 2Δ energy! Inside the cavity, this means that the particle have lost energy, which becomes a photon! Of the right frequency because the cavity is tuned on this frequency!

But: $|E\rangle$ is an energy eigenstate, which is stationary. How do we convince $|E\rangle$ to decay into $|G\rangle$?

If there is an electric field INSIDE THE CAVITY, it could STIMULATE the transition, so $|E\rangle$ of the unperturbed (without field) Hamiltonian is no longer an energy eigenstate, in the cavity, and this must reduce $|E\rangle$ to $|G\rangle$. MASER: Microwave Amplification by Stimulated Emission of Radiation!

At the beginning, due to the little quantum fluctuations there are few photons in the cavity. Then, with resonance, this cavity gets filled soon, and then the molecules start to turn in, and so on...

So, the self-consistency of this field with photons, makes the $|E\rangle \to |G\rangle$ transition happen, and the photon emission as well.

Inside the cavity we assume to have an electric field, so the relevant Hamiltonian is (2). The problem is that $\mathcal{E}$ is going to be a time-dependent field, so something hard to treat. (time-dependent because the field in the cavity is a wave, with frequency $\neq 0$!)

Another problem is that (2) is the Hamiltonian in the $|\uparrow\rangle$, $|\downarrow\rangle$ basis. We prefer to work in the $|E\rangle$, $|G\rangle$ basis!

In the $|E\rangle, |G\rangle$ basis, this can be obtained as follows. Let

$|E\rangle = |1'\rangle$, $|G\rangle = |2'\rangle$ (basis vectors with electric field, "primed")

The H'_{11} element is:

$\langle 1'|H|1'\rangle$ is:

$$= \langle E|H|E\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\right)\begin{bmatrix}E_0+\mu\mathcal{E} & -\Delta \\ -\Delta & E_0-\mu\mathcal{E}\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right) =$$

$$= E_0+\Delta \quad \text{(by doing all the math)}$$

Then, by applying also $\langle 2'|H|2'\rangle$, $\langle 1'|H|2'\rangle$ ..., we obtain:

$$H' = \begin{bmatrix}E_0+\Delta & \mu\mathcal{E} \\ \mu\mathcal{E} & E_0-\Delta\end{bmatrix}$$

The diagonal elements are "drawn": with no $\mathcal{E}$ we have that the energy of $|E\rangle$ is $E_0+\Delta$, and that of $|G\rangle$ is $E_0-\Delta$. The "$\mu\mathcal{E}$" terms appear on the off-diagonal terms.

The general state, in this basis, is Ψ, and is the amplitude for a state to be in E and/or in G:

$$\Psi(t) = \begin{bmatrix}c_E(t)\\ c_G(t)\end{bmatrix} \qquad \text{amplitudes in E and G : } c_E(t),\ c_G(t)$$

So, in this $|E\rangle, |G\rangle$ basis, our Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}c_E(t)\\ c_G(t)\end{bmatrix} = \begin{bmatrix}E_0+\Delta & \mu\mathcal{E} \\ \mu\mathcal{E} & E_0-\Delta\end{bmatrix}\begin{bmatrix}c_E(t)\\ c_G(t)\end{bmatrix}$$

Notice that E_0 is totally irrelevant: it is just entering as a constant diagonal term! We can set it to 0 without any physical problem!

$$\rightarrow i\hbar\frac{\partial}{\partial t}\begin{bmatrix}c_E(t)\\ c_G(t)\end{bmatrix} = \begin{bmatrix}\Delta & \mu\mathcal{E} \\ \mu\mathcal{E} & -\Delta\end{bmatrix}\begin{bmatrix}c_E(t)\\ c_G(t)\end{bmatrix} \qquad (3)$$

$\dot{\Psi}$ = H Ψ

The real difficulty here is that $\mathcal{E}$ is a function of time!!!

The approach to solve this equation is: try to solve it as if $\mathcal{E}$ was NOT time-dependent. So, we would have:

$$\begin{bmatrix}c_E(t)\\ c_G(t)\end{bmatrix} = \begin{bmatrix}e^{-i\frac{\Delta}{\hbar}t} & \\ & e^{+i\frac{\Delta}{\hbar}t}\end{bmatrix} \qquad (4)$$

(with $\mathcal{E}=0$ this is the solution!!!)

However, this is NOT the solution of (3)! Because, $\mathcal{E}\neq 0$! So, what we can do is to CORRECT it: (4) is rewritten as

$$\begin{bmatrix}c_E(t)\\ c_G(t)\end{bmatrix} = \begin{bmatrix}e^{-i\frac{\Delta}{\hbar}t} & \beta_E(t) \\ \beta_G(t) & e^{+i\frac{\Delta}{\hbar}t}\end{bmatrix} \qquad (5)$$

We can use (5) as an ansatz, plug it in (3), and see what happens. It would not be surprising to have something not containing Δ, since it is already in the exponentials!

After some algebra, we obtain:

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}\beta_E(t)\\ \beta_G(t)\end{bmatrix} = \begin{bmatrix}0 & e^{i\omega_0 t}\mu\mathcal{E} \\ e^{-i\omega_0 t}\mu\mathcal{E} & 0\end{bmatrix}\begin{bmatrix}\beta_E(t)\\ \beta_G(t)\end{bmatrix} \qquad (6)$$

where ω_0 is the Larmor frequency or, in other words,

$$\omega_0 = \frac{2\Delta}{\hbar}$$

Let's recall what we are doing: instead of c_E and c_G we are computing β_E and β_G, which are "equivalent" due to (5)!

And, $\mathcal{E}$ is still time-dependent! In particular, if operation is off-resonant (the cavity is tuned),

$\mathcal{E}(t) = 2\mathcal{E}_0 \cos(\omega_0 t)$ ω_0 of the cavity = Larmor frequency!

$= \mathcal{E}_0 \left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$

So, we can substitute in (6), leading to:

$$\rightarrow i\frac{\partial \beta_E}{\partial t} = \frac{\mu \mathcal{E}_0}{\hbar}\left(1 + e^{2i\omega_0 t}\right)\beta_G(t) \qquad (7)$$

$$i\frac{\partial \beta_G}{\partial t} = \frac{\mu \mathcal{E}_0}{\hbar}\left(1 + e^{-2i\omega_0 t}\right)\beta_E(t)$$

Now: what we would like is to forget about the $\left(1+e^{\pm 2i\omega_0 t}\right)$ terms. In fact, $\mu\mathcal{E}_0$ is a very small number. $\mu\mathcal{E}_0$ is an energy which, compared to the energy scales of the system, is small. In particular,

$\mu\mathcal{E}_0 \ll \delta$!

Since δ is related to ω_0, with

$\hbar\omega_0 = 2\delta$,

$\mu\mathcal{E}$ is very, very smaller than $\hbar\omega_0$; this means, looking (7), that, independently of what is inside the parentheses (which might be from 0 to 2), $\frac{d\beta_E}{dt}$ and $\frac{d\beta_G}{dt}$ are going to be small.

Even if $e^{i\omega_0 t}$ oscillates quickly, since $\frac{d\beta_E}{dt}$ and $\frac{d\beta_G}{dt}$ are small, β_E and β_G do not change very much. So, even if $e^{i\omega_0 t}$ is quickly varying, it doesn't contribute significantly to the differential equation

So, we can ignore the $e^{\pm i2\omega_0 t}$! And we obtain:

$$i\frac{\partial \beta_E}{\partial t} \approx \frac{\mu \mathcal{E}_0}{\hbar}\beta_G(t) \qquad (8)$$

$$i\frac{\partial \beta_G}{\partial t} \approx \frac{\mu \mathcal{E}_0}{\hbar}\beta_E(t)$$

which are much better! We can now differentiate (8a) and multiply by i:

$$i\,i\frac{\partial^2 \beta_E}{\partial t^2} = \mu\frac{\mathcal{E}_0}{\hbar}\, i\frac{\partial \beta_G(t)}{\partial t} \overset{(8b)}{=} \left(\mu\frac{\mathcal{E}_0}{\hbar}\right)^2 \beta_E(t)$$

So,

$$\frac{\partial^2 \beta_E}{\partial t} = -\left(\frac{\mu\mathcal{E}_0}{\hbar}\right)\beta_E,$$

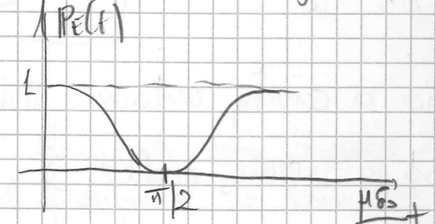
which is a wave equation with energy $\mu\mathcal{E}_0$ instead of Δ, so much smaller frequency than ω_0!

Now, we can write the solution:

$$\beta_E(t) = \cos\left(\frac{\mu\mathcal{E}_0 t}{\hbar}\right), \qquad \text{and the probability to be in } |E\rangle \text{ state is}$$

$$|P_E(t)| = \cos^2\left(\frac{\mu\mathcal{E}_0 t}{\hbar}\right)$$

(it would be $|c_E(t)|^2$, but it ends up, see (5), to be just $|\beta_E|^2$;-)



This means:

Let T the time such that

$$\frac{\mu\mathcal{E}_0}{\hbar}T = \frac{\pi}{2}, \quad \text{or} \quad T = \frac{\pi\hbar}{\mu\mathcal{E}_0}$$

at these times, $|P_E| = 0$, so the system MUST be in the ground state! (if we are not in $|E\rangle$, then we are in $|G\rangle$;-))

So, the problem is now reduced to: we have to design the cavity in such a way that traveling causes the system to have $|P_E| = 0$. We need then the right molecule velocity. If this happens, each input $|E\rangle$ state is converted in a $|G\rangle$ state, releases a photon with the "right" wavelength, and contributes to the electric field $\mathcal{E}(t)$!

So, just by feeding the cavity with excited nitrogen molecules we obtain the MASER effect. Ammonia maser, 1953. Masers do the most perfect

amplification consistent with the uncertainty principle. In fact, the output molecule is uncharged, so it doesn't affect the electromagnetic field of the cavity! No charge → no field perturbation!

The cavity field stimulates the transition, which generates the photon, and which excites the field itself!

In his Nobel lecture, Charles Townes discussed the uncertainty.

Since (we studied)

$$\Delta N \, \Delta\phi \geq \frac{1}{2},$$

which is saturated for coherent states,

if we remember that in a coherent state $|\alpha\rangle$

$$\langle \hat{N} \rangle_{|\alpha\rangle} = |\alpha|^2 \equiv n,$$

$$\Delta N = |\alpha| = \sqrt{n},$$

and, so, for the maser,

$$\Delta N \, \Delta\phi = \frac{1}{2} \quad \Rightarrow \quad \boxed{\Delta\phi = \frac{1}{2\Delta N} = \frac{1}{2\sqrt{n}}}.$$

Since a cavity contains reasonably 10^{12} to 10^{15} photons (which is "n"), $\Delta\phi$ is veeeery small :-)

Th 10 min

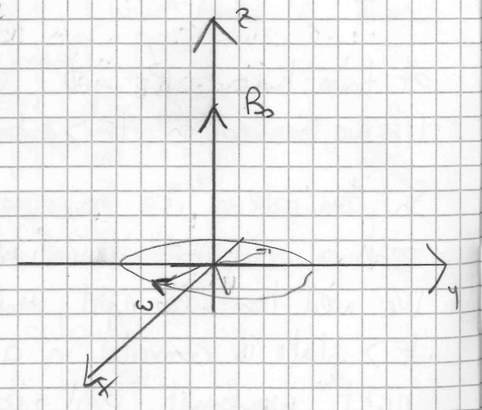
NMR: Nuclear Magnetic Resonance

This is another time-dependent problem!

This starts from having a magnetic field with a big component $B_0 \hat{z}$, plus some magnetic field rotating in the xy plane, with some frequency ω. So, the total magnetic field is

$$\vec{B} = B_0 \hat{z} + B_1 (\cos\omega t \, \hat{x} - \sin\omega t \, \hat{y})$$

Our purpose is to figure out what spins do in such a field!



Since the magnetic field is time-dependent, we risk to end up with a time-dependent Hamiltonian! :-|

So: in Schrödinger picture, the spin Hamiltonian is

$$H_S(t) = -\gamma \vec{B}(t) \cdot \hat{\vec{S}} = -\gamma \left[B_0 \hat{S}_z + B_1 (\hat{S}_x \cos\omega t - \hat{S}_y \sin\omega t) \right], \quad \left(\begin{matrix} \text{spin} \\ \text{Hamiltonian} \end{matrix} \right)$$

of course, time-dependent :-|

Moreover, this is very bad, since at different times it doesn't commute, so time evolution is rather complicated. For instance, at $t = 0$ we have

$$H_S(0) = -\gamma \left[B_0 \hat{S}_z + B_1 \hat{S}_x \right],$$

but at a later time we will have $\hat{S}_z$ and $\hat{S}_y$:-|

No cookbook recipe :-|

Let us try to get the main intuition going for this problem. Without being too formal for now, if we have a state $|\psi\rangle$,

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle,$$

we would like to change the Hamiltonian, to study the same physical problem. This is not trivial: if the physical problem is the same, the Hamiltonian cannot be too different!

What we can try to do is to change the states by an unitary transformation U, and hope that this transformation could simplify the Hamiltonian. So:

$$|\psi'\rangle = U |\psi\rangle$$

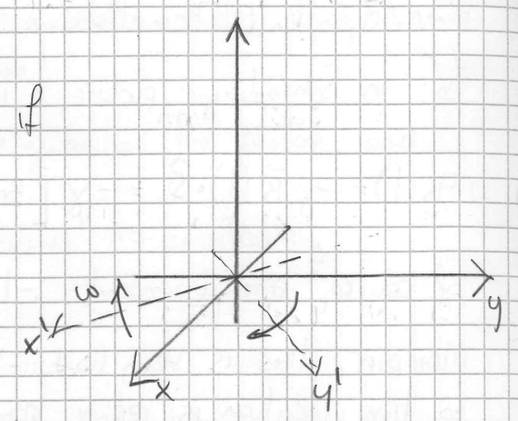
$|\psi'\rangle$: new states; U: unitary transf.; $|\psi\rangle$: old states

A unitary transformation in general is just like a change of basis. To enrich it, U must feature some time dependence, in which case the Hamiltonian could have some strong variation.

This is our only chance to end up with a time-independent Hamiltonian!

$\longrightarrow U^\dagger(t) H_S U(t)$

Let us start from something "intuitive": if we have a system whose Hamiltonian is $\mathbb{0}$, nothing happens is such system.



What would happen if, with $H = \mathbb{0}$ we would have the axes x and y rotating, with angular velocity ω?

Even if our spins are static, for $H_S = \mathbb{0}$, the reference system, the "frame", is rotating, so from the perspective of this frame, spins are rotating!!! And more specifically, they are precessing around the z-axis!

(H_S static)

This means that, from the point of view of the rotating frame, there must be some Hamiltonian, even though there's no Hamiltonian in the static frame! So, which is the Hamiltonian in the rotating frame, H_R?
Well, it is the Hamiltonian such that it rotates the spin with angular velocity ω around $\hat{z}$. This is done by the unitary operator

$$U(t) = e^{-i\omega t \frac{\hat{S}_z}{\hbar}}$$

this is the operator which rotates spins around the $\hat{z}$ direction !!!

So,

$H_R = \omega \hat{S}_z$! Even though $H_S = \mathbb{0}$ in the static frame!

This was the idea/intuition. We will have something else to do....

To obtain the "rotating wavefunction" $|\psi_R\rangle$, we will need

$|\psi_R\rangle = U |\psi\rangle$, being $|\psi\rangle$ the "physical", standing wavefunction.

If we know the Schrödinger equation for $|\psi\rangle$, the one for $|\psi_R\rangle$ will be:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle,$$

$$\hookrightarrow i\hbar \frac{\partial}{\partial t} |\psi_R\rangle = i\hbar \frac{\partial}{\partial t} U |\psi\rangle = \text{two terms:}$$

$$= i\hbar \frac{\partial U}{\partial t} |\psi\rangle + i\hbar U \frac{\partial |\psi\rangle}{\partial t}.$$

For the first term,

$$i\hbar \frac{\partial U}{\partial t} |\psi\rangle = i\hbar \frac{\partial U}{\partial t} U^\dagger U |\psi\rangle = i\hbar \frac{\partial U}{\partial t} |\psi_R\rangle.$$

For the second term,

$$i\hbar U \frac{\partial}{\partial t} |\psi\rangle = U H_S |\psi\rangle \; !$$

$$= U H_S U^\dagger U |\psi\rangle = U H_S U^\dagger |\psi_R\rangle$$

So, we have a new Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi_R\rangle = \underbrace{\left(U H_S U^\dagger + i\hbar \frac{\partial U}{\partial t} U^\dagger \right)}_{H_R} |\psi_R\rangle \; !$$

This is the "rotating Hamiltonian"! Hopefully, it's going to be simpler! Notice that it contains two terms: a "similarity transformation", plus an extra term.
This is compatible with the $H_S = \mathbb{0}$ possibility: there is still a 2nd part ;-)

U is such that

$$i\hbar \frac{\partial U}{\partial t} U^\dagger = \omega \hat{S}_z,$$

and, for this, U is nothing but

$$U = e^{-i\omega t \frac{\hat{S}_z}{\hbar}},$$ which is what we've written before!

To close the discussion, we've found that there's a new Hamiltonian,

$\hat{H}_R = U H_S U^\dagger + i\hbar \frac{\partial U}{\partial t} U^\dagger$, $U = e^{-\frac{i\omega t}{\hbar} \hat{S}_z}$,

and $|\psi_I\rangle = e^{\frac{i\omega t \hat{S}_z}{\hbar}} |\psi_{R,I}\rangle$

This came because we said that $|\psi_S\rangle$ was going to be $U|\psi\rangle$.

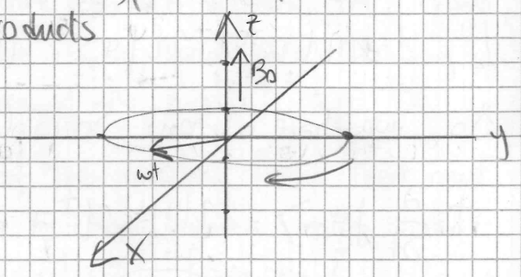
Now our problem is in $|\psi_R\rangle$, with Hamiltonian $\hat{H}_R$, where

$$i\hbar\frac{\partial U}{\partial t}U^\dagger = \omega\hat{S}_z \quad \text{(very simple)}.$$

Now: is U as $\psi^\dagger$ time-independent?

Hopefully. See next time ;-)

Lecture 18: Two State Systems (continued), Multiparticle states and tensor products



We were talking about NMR.

We have a large magnetic field B_0, along z (longitudinal), and a rotating field on the xy plane, clockwise:

$$\vec{B}(t) = B_0\hat{z} + B_1\left(\cos\omega t\,\hat{x} - \sin\omega t\,\hat{y}\right)$$

Typically, in the problems of interest, $B_1 \ll B_0$!

Then, the spin Hamiltonian is given by a constant γ, that relates the dipole moment of a particle to its spin angular momentum,

$$H_S(t) = -\gamma\vec{B}(t)\cdot\vec{S} = -\gamma\left(B_0\hat{S}_z + B_1\left(\cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y\right)\right)$$

This is clearly time-dependent, and at different times it doesn't commute! :C

Since we have no convenient commutation relation, only the time-ordered exponential could be the only chance :C

The idea is to use the ROTATING FRAME (i.e.: interaction picture!): a frame which rotates with the magnetic field in such a way to simplify the physics (hopefully!). In particular, we would like to have a time-independent Hamiltonian.

So we have to understand how the Hamiltonian looks when we "look at it" from another frame: transformation of Hamiltonians.

Imagine that our system has no magnetic field; in this case, a spin state would not precess.

BUT then, let's imagine that, in this situation, we jump in a rotating frame: from OUR perspective spins move! Precess! Even if we don't have a magnetic field! So, from this "rotating Hamiltonian perspective, $\hat{H}_R$, it's like having a $\vec{B}$ field!

So, we can imagine that: if there is a magnetic field, and we choose a rotating frame "solidal", "coherent" with it, then spins appear as STATIC!

So, let's consider an evolution operator U as

$$U(t) = \exp\left(-\frac{i\omega t}{\hbar}\hat{S}_z\right)(a), \text{ where } \omega t \text{ is an } \underline{\text{angle}}$$

$U(t)$ rotates states, and, for a time-independent system, the unitary operator is

$$U = \exp\left(-i\frac{\hat{H}}{\hbar}t\right), \text{ so, if we perform identification our Hamiltonian } \hat{H}_W \text{ is}$$

$$\hat{H}_W = \omega\hat{S}_z.$$

Usually, we know that, given an unitary evolution operator U, the Hamiltonian is obtained as:

$$\hat{H} = i\hbar\left(\frac{\partial U}{\partial t}\right)U^\dagger. \quad (1)$$

Now! Let's try to do what follows: act with the evolution operator on $|\Psi,t\rangle$: a complicated state, including time dependence and very hard to be studied. We define the rotated eigenfunction as this thing:

$$|\psi_R,t\rangle = U(t)|\psi_I,t\rangle \quad (2) \text{ this is a DEFINITION}$$

And we hope that Schrödinger equation for the left hand side is simple.

Let's do the computation of (2). [NOTICE: Ψ_R, NOT Ψ_R!]

$$|\Psi_R, t\rangle = U(t)\,|\Psi, t\rangle = U(t)\,U_S(t)\,|\Psi, 0\rangle, \quad (2)$$

where $U_S(t)$ is the evolution operator of the spin system: the unitary operator which transforms $|\Psi, 0\rangle$ into $|\Psi, t\rangle$!
So, $U(t)\,U_S(t)$ is the total evolution operator evolving the state.
Recalling (1),

$$H_R = i\hbar\left(\frac{\partial}{\partial t}\left[U(t)\,U_S(t)\right]\right)\left[U_S^\dagger(t)\,U^\dagger(t)\right] \qquad \left(\text{since } (U\,U_S)^\dagger = U_S^\dagger\,U^\dagger\right)$$

Now, let's compute the derivative:

$$= i\hbar\left(\frac{\partial U}{\partial t}\right)U^\dagger + i\hbar\,U(t)\left(\frac{\partial U_S(t)}{\partial t}\right)U_S^\dagger\,U^\dagger(t)$$

$U_S\,U_S^\dagger = I$!

$$= U(t)\,\hat{H}_S\,U^\dagger(t)$$

Notice that the first term is exactly H_U, Hamiltonian associated to U, plus the second one!

$$\Rightarrow H_R = H_U + U(t)\,\hat{H}_S\,U^\dagger(t) \quad (4)$$

We already derived something like this!
So, in the rotating frame the Hamiltonian is the transformed old Hamiltonian, H_S, plus the Hamiltonian associated to the time evolution operator (U) of the rotating frame!

More explicitly, by calling (3),

$$H_R = \omega\hat{S}_z + \exp\left(-\frac{i\omega t S_z}{\hbar}\right)\left[-\gamma\left(B_0\hat{S}_z + B_1\left(\cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y\right)\right)\right]\wedge \exp\left(\frac{i\omega t S_z}{\hbar}\right)$$

This is our new Hamiltonian and, unless it simplifies, we have not gained so much :-|
Before this, it is to be remarked that the down between (3) and (4) is that

$$i\hbar\frac{\partial}{\partial t}|\Psi_R, t\rangle = \hat{H}_R\,|\Psi_R, t\rangle \quad [\text{did last time!}]$$

Let's simplify this thing.
The exponentials contain $\hat{S}_z$, so they commute with the $\hat{S}_z$ in the parentheses:

$$\rightarrow \hat{H}_R = (\omega - \gamma B_0)\hat{S}_z - \gamma B_1\, e^{-i\frac{\omega t \hat{S}_z}{\hbar}}\left(\cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y\right)e^{+i\frac{\omega t \hat{S}_z}{\hbar}}$$

Now: we could either expand the exponentials and multiply, or use the formula like

$$e^{A}\,B\,e^{-A} \text{ and simplify.}$$

Now, let's look at this from a different perspective.
We need to know $\hat{H}_R$, and to simplify it exploiting the rotating frame.
Let's call $M(t)$ the thing

$$M(t) = e^{-i\frac{\omega t \hat{S}_z}{\hbar}}\left(\cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y\right)e^{+i\frac{\omega t}{\hbar}\hat{S}_z}$$

and take it's time derivative: maybe, from this, we can see some differential equation or something helpful!

$$\frac{\partial M(t)}{\partial t} = e^{-i\frac{\omega t}{\hbar}S_z}\left(\left[-i\frac{\omega}{\hbar}\hat{S}_z, \left(\cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y\right)\right] + \right.$$

(↳ just like in coherent or squeezed states, we derive the exponential, it causes an $\hat{S}_z$ to drop, and ...)

$$\left. -\omega\sin\omega t\,\hat{S}_x - \omega\cos\omega t\,\hat{S}_y\right)e^{i\frac{\omega t}{\hbar}\hat{S}_z}$$

Since $\left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$, $\left[\hat{S}_z, \hat{S}_y\right] = -i\hbar\hat{S}_x$,

$$= e^{-i\frac{\omega t \hat{S}_z}{\hbar}}\left(-\frac{i\omega}{\hbar}\left(i\hbar\hat{S}_y\cos\omega t + i\hbar\hat{S}_x\sin\omega t\right) - \omega\sin\omega t\,S_x - \omega\cos\omega t\,S_y\right) = e^{i\frac{\omega t \hat{S}_z}{\hbar}}$$

$$= 0$$

So, $\frac{\partial M(t)}{\partial t} = 0$!!! So, $M(t)$ is a CONSTANT: has NO time dependence!

And, since it is constant, we can evaluate it at any time we want, and it will be constant.
So, go for $t=0$, and

$$\underline{M(0) = N(1) = \hat{S}_x} \quad !$$

So, our whole Hamiltonian is

$$\hat{H}_R = \left(-\gamma B_0 + \omega\right)\hat{S}_z - \gamma B_1 \hat{S}_x$$

Very nice! Two pieces!

And the only rotating part seen from the rotating frame is

$$-\gamma B_1 \hat{S}_x \quad !$$

Recalling that $\omega_0 = \gamma B_0$ is the Larmor frequency associated to B_0,

$$H_R = -\gamma\left[\left(B_0 - \frac{\omega}{\gamma}\right)\hat{S}_z + B_1 \hat{S}_x\right] = -\gamma\left[B_0\left(1 - \frac{\omega}{\omega_0}\right)\hat{z} + B_1 \hat{x}\right] =$$

$$= -\gamma \cdot \vec{B}_R \cdot \vec{S},$$

by using the usual spin notation, with $\vec{B}_R$ the "rotating $\vec{B}$", even if now it doesn't rotate anymore!

$$\vec{B}_R = B_0\left(1 - \frac{\omega}{\omega_0}\right)\hat{S}_z + B_1 \hat{S}_x$$

So, finally, we can say that (from (3))

$$|\Psi, t\rangle = U^+ |\Psi_R, t\rangle = e^{i\frac{\omega t \hat{S}_z}{\hbar}} e^{-i\frac{H_R t}{\hbar}} |\Psi, 0\rangle$$

because the $\hat{H}_R$ proves to be time-independent and then in the rotating frame we do not longer need the time-evolution operator!

$$= e^{i\frac{\omega t \hat{S}_z}{\hbar}} e^{+\frac{i\gamma \vec{B}_R \cdot \vec{S} t}{\hbar}} |\Psi, 0\rangle \quad (4)$$

Notice that, at $t=0$, $|\Psi_R, 0\rangle = |\Psi, 0\rangle$!

(4) is the complete solution of the problem of the rotating spins!

Notice that we used the formula

$$\frac{d}{dt}\left[e^{tA} B\, e^{-tA}\right] = e^{tA}\left[A, B\right]e^{-tA}$$

Ok. So. We simplified the formula, but it's like... Just math! What does this formula describe?!

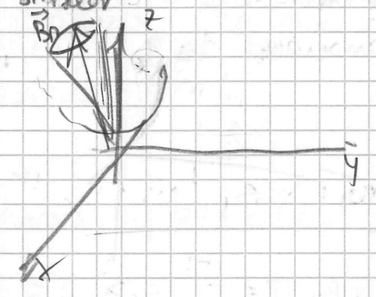
- Applications always have $B_1 \ll B_0$.

Given this, let us study some cases

- $\omega \ll \omega_0$

In this case, the magnetic field rotates much slower

$$\vec{B}_R \approx B_0 \hat{z} + B_1 \hat{x}$$



Most of the field is along $\hat{z}$

Assuming that at $t=0$ the spin is up, along $+\hat{z}$,

the magnetic field rotates the spin states around the $\vec{B}_R$ axis, around such cone defined by $\vec{B}_R$.
Still, there is also the $\exp(i\frac{\omega t \hat{S}_z}{\hbar})$ but, being $\omega \ll \omega_0$, with ω_0 the very fast rotation induced by $\vec{B}_R$, the cone itself precesses around $\hat{z}$, very slowly!

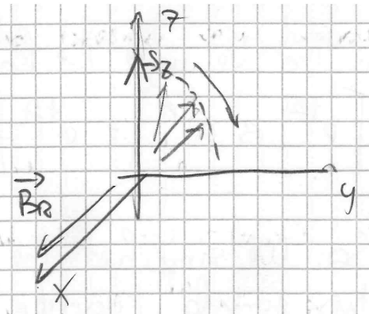
- The important one is $\boxed{\omega = \omega_0}$. In such case, what is a resonance,

$$\vec{B}_R = B_1 \hat{x}$$

So the radiofrequency ω_0 is chosen coherently with the spins! We lose the $\hat{z}$ component, and $\vec{B}_R$ is just $B_1 \hat{x}$!

So,

and spins rotate, with B_1, on the yz plane, with a frequency much smaller than ω_0.

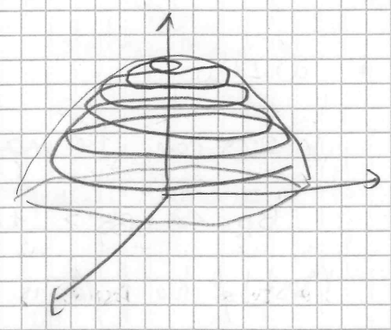


What is the second term, the

$$\exp\left(i\omega \frac{t}{\hbar} \hat{S}_z\right)$$ goes to 0?

It rotates around z! A combined zy and z axis rotation very fast!

A SPIRAL!



If we choose the Larmor frequency associated to B_1, ω_1, to be

$\omega_1 T = \frac{\pi}{2}$,

$\omega_1 = \gamma B_1$,

$\gamma B_1 T = \frac{\pi}{2} \Rightarrow T = \frac{\pi}{2\gamma B_1}$ is called "90° pulse"

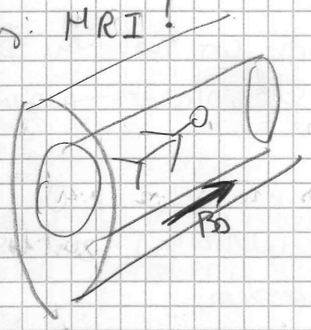
If we keep B_1 be active enough, B_1 disappears but the other magnetic field allows to keep the guy rotating!

This is basically the Magnetic Resonance Imaging: MRI!

This is much better than x-rays!

2 Tesla magnet, cooled with Liquid Helium.

Basically, this allows to image the local concentration of water: the magnetic field reacts with hydrogen, with the protons. Protons have magnetic dipole moments which get aligned to the B_0. Due to temperature, 1 proton over 1E6 align, but that's enough. They send the 90° pulse, the spin starts, we reach the 90°, the dipole rotates like crazy, and this is basically a generator of EM waves, and detectors pick up this signal. The strength of this is



proportional to the concentration of water, and then to distinguish different tissues.

Because of interactions with other spins, there is a relaxation time T_2, such that we lose rotation, and another T_1 such that we realign to the magnetic field. The measurements concern T_1 and T_2.

For any liquid you can measure a T_1 and T_2.

The noise is the gradient magnets that move! The B_0 is changed locally, so ω_0 and spin velocity change as well! And get very high, 0.5 mm, resolutions.

Multiparticle states and Tensor products 54 min

Idea! we have a system with more than a particle.

If we have 2 particles for instance, for

Particle 1: we can describe its quantum mechanics by a complex vector space V, operators $\hat{T}_1, \hat{T}_2 \dots$

Particle 2: complex vector space W, operators $\hat{S}_1, \hat{S}_2 \dots$

We studied several operators in quantum mechanics: momentum, position, or even 3 position / 3 momenta (in 3D space), angular momentum, spin, Hamiltonians... And we have these, for each particle.

Question: how do we describe the COMPOSITE system, i.e., the system of the two particles, that could also interact!?

A description of particle 1 is a vector $v \in V$

particle 2 is a vector $w \in W$.

So, we could list v and w, (v, w), and thus could be the information about each particle.

This is kind of oversimplified: it is better to use a NOTATION encoding (v, w) AS

$v \otimes w$, where $\otimes$ is the tensor product

This is to somehow say that we're going to do more than just listing v and w!

So, $\otimes$ is indicating that V and W are two, together, and we must deal with this togetherness.

$v \otimes w$ is said to be an element of a new, complex vector space,

$V \otimes W$: the tensor product of V and W.

Now, let's discuss the properties of this object. If we have

$(\lambda v) \otimes w$

is it somehow related to $v \otimes w$?

Well, yes. This is a tensor product. Shankar calls "$\otimes$" a direct product, but mathematically this is wrong.

We say that λ can go out:

$$(\lambda v) \otimes w = \lambda (v \otimes w) = v \times (\lambda w)$$

So, $\otimes$ allows numbers to go out! Without any complex conjugate! We impose this property.

Other property. If we have $v_1 \in V_1 \quad v_2 \in V_1$

$w_1 \in W_1 \quad w_2 \in W_1$

$v_1 \otimes w_1 \in V \otimes W_1 \quad v_2 \otimes w_2 \in V \otimes W_1$

then, $V \otimes W$ is supposed to be a linear vector space, so it should be true that:

$$\alpha (v_1 \otimes w_1) + \beta (v_2 \otimes w_2) \in V \otimes W$$

This could give us some intuition about why simply listing the vectors is naive: we can combine, SUPERIMPOSE states! You cannot just know the state of one particle and of the other separately: what they are doing together is important!

The most general state of this system is a superposition in which particle 1 does something, part. 2 something, plus part. 1 does something else and part. 2 something else.

This is the origin of the idea of entanglement. You can't know what particle 1 does without knowing what is doing particle 2, and vice versa.

There is still something else to discuss about this tensor product. Suppose we have

$(v_1 + v_2) \otimes w_1$

we have to declare that this is

$$= v_1 \otimes w_1 + v_2 \otimes w_1$$

So, the composite system should be thought as a superposition of the first state can be v_1 or v_2, when the other just in w_1, and this is a superpostion of $v_1 \otimes w_1$ plus $v_2 \otimes w_1$

(Another axiom of the tensor product!

Similarly,

$$v_1 \otimes (w_1 + w_2) = v_1 \otimes w_1 + v_2 \otimes w_2$$

This defines completely how to calculate with this 2-particle Hilbert space and how to define the states.

Let us try to add some more intuition into this. The space $V \otimes W$ is the space spanned by things like $v_i \otimes w_i$. So, all the rules that we just mentioned allow to put everything together.

Let $(e_1 \dots e_n)$ a basis for V_1

$(f_1 \dots f_m)$ a basis for W_1

then the set of things like

$e_i \otimes f_j$, which are $n \times m$ vectors,

$i = 1 \dots n$,

$j = 1 \dots m$

form a basis for $V \otimes W$!!!

Basically, if we have a 4-dimensional space V and a 10-dimensional W, $V \otimes W$ will be 40-dimensional!

This is a consequence of these superpositions and linearity rules!

So,

$$\dim \{V \otimes W\} = \dim \{V\} \times \dim \{W\}$$

These ideas are more subtle than what they look.

Operators on $V \otimes W$

We need to define operators on $V \otimes W$.
Suppose $\hat{T} \in \mathcal{L}(V)$, and $\hat{S} \in \mathcal{L}(W)$ (are operators on V and W, respectively).
Then, we define the operator

$$\hat{T} \otimes \hat{S}$$

which is an operator on $V \otimes W$:

$$\hat{T} \otimes \hat{S} \in \mathcal{L}(V \otimes W)$$

If this is an operator, then we should know how it acts on each element $v \otimes w \in V \times W$:

$$\hat{T} \otimes \hat{S} \,(v \otimes w) = (\hat{T} v) \otimes (\hat{S} w)$$

This is a definition, but also a matter of common sense: which other definition could be possible?
In fact, $\hat{T} v$ is a vector on V, while $\hat{S} w$ is a vector on W!
So, each operator acts on its thing!

This is fine. The operators that instead could be more surprising are: suppose we have $\hat{T}_1 \in \mathcal{L}(V)$, and we want it to act on the tensor product.
Well, we need another operator. This need is called "upgrading" an operator that acts on V to an operator that acts on the whole thing.
What we do is the following:

$$\hat{T}_1 \in \mathcal{L}(V) \implies \hat{T}_1 \otimes \mathbb{1} \in \mathcal{L}(V \otimes W)$$

So, we multiply it times the identity!
Instead, if we have

$$\hat{S}_1 \in \mathcal{L}(W), \implies \mathbb{1} \otimes \hat{S}_1 \in \mathcal{L}(V \otimes W)$$

Now, question: do these two operators commute?
Let's see:

$$(\hat{T}_1 \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes \hat{S}_1)\,(v \otimes w) = (T_1 \cdot 1)(v \times \hat{S}_1 w) = (\hat{T}_1 v) \times (\hat{S}_1 w)$$

is the same as ?

$$(\mathbb{1} \otimes \hat{S}_1) \cdot (\hat{T}_1 \otimes \mathbb{1})\,(v \times w) = (1 \cdot \hat{T}_1 v) \otimes (\hat{S}_1 w)$$

Yes! They commute!
So, these two operators, which originated from single particle operators, commute! They "don't know about each other"!
A good example of this thing is: when we try to compute the Hamiltonian of the whole system, it is:

$$\hat{H}_T = \hat{H}_1 \otimes \mathbb{1} + \mathbb{1} \times \hat{H}_2$$

$\hat{H}_1$ and $\hat{H}_2$ being the Hamiltonians of the first and second single-particle systems!

Example

A famous example: two spin-$\frac{1}{2}$ particles!

The first particle has a state $|+\rangle_1$, $|-\rangle_1$,
the second has $|+\rangle_2$, $|-\rangle_2$;

How do we form the tensor product?
We are supposed to take the tensor product.
So, the tensor vector space is spanned by a vector in the first times a vector in the second:

$$|+\rangle_1 \otimes |+\rangle_2 \;,\; |+\rangle_1 \otimes |-\rangle_2 \;,\; |-\rangle_1 \otimes |+\rangle_2 \;,\; |-\rangle_1 \otimes |-\rangle_2$$

These are the four possibilities.
So, two spin states form a four-dimensional complex vector space!
And the most general state is the following:

$$|\psi\rangle = \alpha_1 |+\rangle_1 \otimes |+\rangle_2 + \alpha_2 |+\rangle_1 \otimes |-\rangle_2 + \alpha_3 |-\rangle_1 \otimes |+\rangle_2 + \alpha_4 |-\rangle_1 \otimes |-\rangle_2 .$$

Now, let us try to understand which is the result of acting with the total z component of the spin on this state: $\hat{S}_z^T$

$$\hat{S}_z^T |\psi\rangle$$

First of all, let us recall that the total Hamiltonian should be written as

$$\hat{S}_z^T = \hat{S}_z^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z^{(2)}$$

(upgrade of $\hat{S}_z$! on the two spaces)

What is doing this? Well,

$$(\hat{S}_z^{(1)} \otimes \mathbb{1})\,|\psi\rangle = \alpha_1 \hat{S}_z^{(1)} |+\rangle_1 \otimes |+\rangle_2 + \alpha_2 \hat{S}_z^{(1)} |+\rangle_1 \otimes |-\rangle_2 +$$
$$+ \alpha_3 \hat{S}_z^{(1)} |-\rangle_1 \otimes |+\rangle_2 + \alpha_4 \hat{S}_z^{(1)} |-\rangle_1 \otimes |-\rangle_2 .$$

But we know that

$\hat{S}_z^{(1)}|+\rangle_1 = \frac{\hbar}{2}|+\rangle_1$ so we have

$= \frac{\hbar}{2}\left(\alpha_1|+\rangle_1\otimes|+\rangle_1 + \alpha_2|+\rangle_1\otimes|-\rangle_2 \underbrace{-}\alpha_3|-\rangle_1\otimes|+\rangle_2 - \alpha_4|-\rangle_1\otimes|-\rangle_2\right)$

Instead,

$(\mathbb{1}\times S_z^{(2)})|\psi\rangle =$ (similarly)

$= \frac{\hbar}{2}\left(\alpha_1|+\rangle_1\otimes|+\rangle_2 - \alpha_2|+\rangle_1\otimes|-\rangle_2 + \alpha_3|-\rangle_1\otimes|+\rangle_2 - \alpha_4|-\rangle_1\otimes|-\rangle_2\right)$

So when we add these two pieces,

$\hat{S}_z^T|\psi\rangle = \frac{\hbar}{2}\left(\alpha_1|+\rangle_1\otimes|+\rangle_1 - \alpha_4|-\rangle_1\otimes|-\rangle_2\right)$

So, if we wanted a state with total $\hat{S}_z^T = 0$, then we would have to put $\alpha_1 = \alpha_4 = 0$.

Instead: we could calculate what $\hat{S}_y^T|\psi\rangle$ is, and $\hat{S}_x^T$, and figure out if there is a state with total angular momentum equal 0.

Next lecture: teleportation and Bell's inequalities!

Lecture 19: multiparticle states and tensor product

If we have a quantum mechanical system of 1 particle described by a vector space V and one of another particle with W, the quantum mechanics of the total system composed by the 2 particles is defined on a new vector space, $V\otimes W$

In fact, we can't just know what particle 1 and 2, separately, are doing!

There's something more!

The elements of $V\otimes W$ have form

$\sum_{ij} a_{ij}\, v_i\otimes w_j$: this is a general state in the system

What is the vector 0 in this tensor space? It is:

$0 \in V\otimes W$: it is $0\otimes w_j$, ~ 0 in V

or $v_i\otimes 0$ ~ 0 in W

We see that since we know that

$av\otimes w = a(v\otimes w)$ a is a number,

so if we pose $a = 0$, we have the 0 vector!

Once we put 0 in v or w, we have 0!

Another thing we haven't done last time was to define an inner product on the new tensor vector space!

How could we "get numbers" from two vectors in the tensor space?

Inner product

$\langle \sum_{ij} a_{ij}\, v_i\otimes w_j\,,\, \sum_{pq} b_{pq}\, v_p\otimes w_q\rangle$

If we assume linearity in the right arguments and anti-linearity on the left input, we have:

$\sum_{ij} a_{ij}^* \sum_{pq} b_{pq}\langle v_i\otimes w_j\,,\, v_p\otimes w_q\rangle$ (take the sums out...)

what to do with the remaining inner product?

$= \sum_{ij}\sum_{pq} a_{ij}^* b_{pq} \underbrace{\langle v_i, v_p\rangle_V}_{\text{inner product in } V} \underbrace{\langle w_j, w_q\rangle_W}_{\text{on } W}$

Notice that if any of these terms was 0, then the inner product would be zero ;-) Everything so far looks consistent! ;-)

Now that we have an inner product, we are going to define an entangled state. Then, we will build bases of entangled states, and discuss some examples.

We discussed last time that there is a state of two spin-$\frac{1}{2}$ particles This state was

$|\psi\rangle = \alpha\left(|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle\right)$ [not specifying subscripts 1 and 2: the order is 1 first, 2 later]

Now that we have an inner product, let's try to normalize this state!

How could we do it? Let's take the inner product of this state with itself: $\langle \Psi, \Psi \rangle =$

$$= \alpha^* \alpha \left\langle |+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle \;,\; |+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle \right\rangle \quad (1)$$

We have to expand the "crossed" terms: each one at left with the right one, and we get for instance

$$\left\langle |+\rangle\otimes|-\rangle \;,\; |+\rangle\otimes|-\rangle \right\rangle = \langle |+\rangle, |+\rangle\rangle \langle |-\rangle, |-\rangle\rangle = 1$$

On the other hand,

$$\left\langle |+\rangle\otimes|-\rangle, \; |-\rangle\otimes|+\rangle \right\rangle = 0 \; !$$

Another way of writing is to use bra-ket: (same thing)

$$\left[\langle +|_{(1)} \otimes \langle -|_{(2)} = \langle -|_{(1)} \otimes \langle +|_{(2)} \right] \Big| \; |+\rangle_{(1)} \otimes |-\rangle_{(2)} = |+\rangle_{(1)} \otimes |-\rangle_{(2)} \Big]$$

(with bras we can use the same order :-)

(1)
At the end we have

$$= |\alpha|^2 (1+1) = 2|\alpha|^2,$$

So, to ask

$$2|\alpha|^2 = 1 \implies \alpha = \frac{1}{\sqrt{2}}$$

and

$$\hookrightarrow |\Psi\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle \right]$$

is our normalized state!

This state, which we introduced last lecture and now normalized, is with zero S_z component, zero S_x, zero S_y: it has NO ANGULAR momentum. So it is rotationally-invariant! (homeworks!)

Once that we have this, let's talk about entangled states!

Entangled states

These states are a remarkable example of situation where we can't study just the first or second parts of the tensor product standalone.

We learned that $V \otimes W$ includes state superpositions

$$\sum \alpha_{ij} \, v_i \otimes w_j$$

If somebody hands us a state like this, maybe we could do some trickery and find that it equals some vector $v_* \otimes w_*$, where

v_* and w_* belong to V and W, respectively?

If we could have this, then we would be able to say that, however interesting is $\sum \alpha_{ij} \, v_i \otimes w_j$, it can be studied as V_* and W_*, standalone! Particle 1 in state V_*, particle 2 in W_*. If so, the state of the two particles is NOT ENTANGLED (this is a definition of what entangled states are NOT: if we can factor them, then they are not entangled!)
If such V_* and W_* do not exist, then the state is entangled!

So, it might take some work to figure out if a state is entangled or not! To perform some calculations and get some understanding, we will now focus on a simple two-dimensional vector space: V, W 2-dimensional complex vector spaces

V will have a basis (e_1, e_2)
W will have a basis (f_1, f_2)

So, the most general state we could write is:

$$a_{11} \, e_1 \otimes f_1 + a_{12} \, e_1 \otimes f_2 + a_{21} \, e_2 \otimes f_1 + a_{22} \, e_2 \otimes f_2 \quad (2)$$

The question is: is this thing equal to a vector in the form

$$(a_1 e_1 + a_2 e_2) \otimes (b_1 f_1 + b_2 f_2) \quad ? \quad (3)$$

In other words: are there numbers a_1, a_2, b_1, b_2, such that this exist? This problem is not so complicated. In fact, for such tensor spaces, distributive laws apply, so we can rewrite (3) as

$$(3) = a_1 b_1 \, e_1 \otimes f_1 + a_1 b_2 \, e_1 \otimes f_2 + a_2 b_1 \, e_2 \otimes f_1 + a_2 b_2 \, e_2 \otimes f_2 ,$$

So, we must have

$$\begin{aligned} a_{11} &= a_1 b_1 \\ a_{12} &= a_1 b_2 \\ a_{21} &= a_2 b_1 \\ a_{22} &= a_2 b_2 \end{aligned}$$

If this holds, we have that (just do the math)

$$a_{11} a_{22} - a_{12} a_{21} = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 = 0$$

In other words, if $a_{11} a_{22} - a_{21} a_{12} = 0$

If we define a matrix $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, if its determinant is 0, then the state is not entangled!!! Because the factorization can take place!

In fact, $\det\{\underline{\underline{A}}\} = 0$ guarantees that this system has a solution!

So, $\det\{A\} = 0 \iff$ NOT ENTANGLED!

We can prove it (min 25) that we can first set $a_{11} = 0$, solve, and do it with $a_{12} = 0$, $a_{11} \neq 0$, and move on.

So, it is quite simple to have $\det\{A\} \neq 0$! For instance, if $a_{12} = a_{21} = 0$ and $a_{11} \neq 0$, $a_{22} \neq 0$, the state is entangled!

There are enough ways to build entangled states that we could build a basis for them!

Intuitively: when a state is entangled, there is some "correlation" between what state 1 and 2 are doing.

Bases of entangled states.

Let's base our discussion on spins.

Let V be the state space of spin-$\frac{1}{2}$, W the same.

$V \otimes W$

So, a state, $|\Psi_0\rangle$, is: (drop the subscripts and the tensor products):

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle|+\rangle + |-\rangle|-\rangle \right]$$

$$\left(\text{or even } \frac{1}{\sqrt{2}} \left[|++\rangle + |--\rangle \right] \right)$$

Notice that

$$\langle \Psi_0 | \Psi_0 \rangle = 1$$

This is an entangled state, because the matrix A would be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \quad (\because)$$

So, there is NO way to factor this!!! As $v_* \otimes w_*$!

So, this is AN entangled state. But since $V \otimes W$ is 4-dimensional, we need still 3 more states to build a basis! 32 min

These are:

$$|\Psi_i\rangle \triangleq \left(\mathbb{1} \otimes \sigma_i \right) |\Psi_0\rangle \qquad i = 1, 2, 3$$

σ_i are the Pauli matrices!

For instance,

$$|\Psi_1\rangle = \left(\mathbb{1} \otimes \sigma_1 \right) \frac{1}{\sqrt{2}} \left(|+\rangle|+\rangle + |-\rangle|-\rangle \right) =$$

$$= \frac{1}{\sqrt{2}} \left(|+\rangle \underbrace{\sigma_1 |+\rangle}_{|-\rangle} + |-\rangle \underbrace{\sigma_1 |-\rangle}_{|+\rangle} \right) = \qquad \left(\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \quad (\because)$$

$$= \frac{1}{\sqrt{2}} \left(|+\rangle|-\rangle + |-\rangle|+\rangle \right)$$

Notice that $|\Psi_1\rangle$ is orthogonal with $|\Psi_0\rangle$!!!

And the other ones are:

$$|\Psi_2\rangle = \mathbb{1} \otimes \sigma_2 \, |\Psi_0\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle|-\rangle - |-\rangle|+\rangle \right)$$

$$|\Psi_3\rangle = \mathbb{1} \otimes \sigma_3 \, |\Psi_0\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle|+\rangle - |-\rangle|-\rangle \right)$$

Are these states orthonormal? Well, if we take

$\langle \Psi_0 | \mathbb{1} \otimes \sigma_i | \Psi_0 \rangle = \langle \Psi_0 | \Psi_i \rangle$ this is 0: we could try it by inspection ;-)

Another question which is nontrivial is: how much is the product

$\langle \Psi_i | \Psi_j \rangle$?

$$\langle \Psi_i | \Psi_j \rangle = \langle \Psi_0 | \left(\mathbb{1} \otimes \sigma_i \right)^\dagger \left(\mathbb{1} \otimes \sigma_j \right) | \Psi_0 \rangle =$$

$$= \langle \Psi_0 | \mathbb{1} \otimes \sigma_i^\dagger \sigma_j | \Psi_0 \rangle = \langle \Psi_0 | \mathbb{1} \otimes (\ldots$$

the product of two Pauli matrices in bed gives us an identity, plus...

$\mathbb{I}\,\delta_{ij} + i\epsilon_{ijk}\sigma_k$.

So we get

$\langle \Phi_0 | \mathbb{I} \otimes \left(-\mathbb{I}\,\delta_{ij} + i\epsilon_{ijk}\sigma_k\right) | \Phi_0 \rangle$

So: From left, we have $\delta_{ij}\langle \Phi_0 | \Phi_0 \rangle$

Plus

$i\epsilon_{ijk}\langle \Phi_0 | \Phi_k \rangle = 0$! Because $|\Phi_k\rangle$ is orthogonal to $|\Phi_0\rangle$.

At the end, we have just

$\langle \Phi_i | \Phi_j \rangle = \delta_{ij}$

Therefore, this basis is orthonormal! This is a basis of orthonormal states in the tensor product of two spin-$\frac{1}{2}$ particles!

And all of these basis states are entangled states! In fact they fill different parts of the matrix! In fact,

$$|\Phi_3\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle - |-\rangle|-\rangle\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$|\Phi_2\rangle = \ldots = \begin{bmatrix} & -i \\ i & \end{bmatrix}$$

The determinants are $\neq 0$!

Now, let us record a family of the basis states in the conventional way written as superpositions of entangled states!

So, for instance! what is $|+\rangle|+\rangle$?

$$|+\rangle|+\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle + |\Phi_3\rangle\right)$$

$$|+\rangle|-\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle - i|\Phi_2\rangle\right) \quad (4)$$

$$|-\rangle|+\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle + i|\Phi_2\rangle\right)$$

$$|-\rangle|-\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle - |\Phi_3\rangle\right)$$

by inspection!

There are the simple bases, expressed as superpositions of entangled states!

This $\{|\Phi_0\rangle \ldots |\Phi_3\rangle\}$ is called the Bell basis!

Before moving on, a couple of remarks: two facts

(1) If we have an orthonormal basis, one of the postulates of measurements in quantum mechanics can be stated as saying that we can do an experiment in which we find the probability of our state being along any of these basis states, of the orthonormal basis. With the experiment we can detect the system state, and it will collapse on one of our basis states, with some probability.

So, given an orthonormal basis $|e_1\rangle \ldots |e_n\rangle$, we can measure a state $|\Psi\rangle$ and we get that the probability to be in $|e_i\rangle$ is $|\langle e_i|\Psi\rangle|^2$, and if we perform this measure, $|\Psi\rangle$ will collapse on $|e_i\rangle$.

After the measurement the state goes from into some $|e_k\rangle$.

(2) Pauli matrices are not just Hermitian, but they square to 1, so they are also unitary. So, they can be -realized- as a time evolution Imagine that we have a state, and multiply it for instance for σ_1. This is not just a math exercise: σ_i is a unitary operator, so it could correspond to some time evolution! So, there is some Hamiltonian that we can construct that will evolve the state and obtain the state itself, as if it was multiplied times σ_i!

So, $\sigma_1, \sigma_2, \sigma_3$, are unitary, as operators they can be realized by time evolution, with a suitable Hamiltonian. Talking about spin states, this could be, for example, a magnetic field!

In fact, we could show that

$$e^{i\frac{\pi}{2}(-\mathbb{I}+\sigma_i)} = e^{-i\frac{\pi}{2}} e^{i\frac{\pi}{2}\sigma_i} = (-i)\left(\mathbb{I}\cos\tfrac{\pi}{2} + i\sigma_i \sin\tfrac{\pi}{2}\right)$$

they commute so we can split the exponential

$$= (-i)(i\sigma_i) = \sigma_i$$

So, we've just written σ_i as an exponential of a Hermitian operator, and therefore we could say that

$-\mathbb{I} + \sigma_i = \frac{tH}{\hbar}$, and realize such a system ;-)

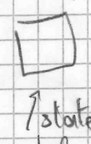
Now ---

Quantum Teleportation

This is something which seems, clamorously, impossible: dematerializing an object and create it somewhere else!

Quantum mechanically, this seems something not so impossible. We're talking about a 1993 discovery! (Bennet @ IBM, Brassard, Crepeau, Josza (all in Montreal), Peres, Wooters).

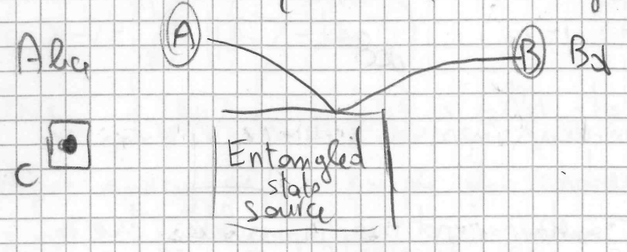
Let's discuss this "game". We have two people involved, "Alice" and "Bob".
Assume that Alice has a quantum state to her, and that it is a spin-$\frac{1}{2}$ particle (this is nice because we have discrete levels):
$\alpha|+\rangle + \beta|-\rangle$
Alice's goal is to send this state to Bob, who is far away.
Alice's state is not entangled or whatever.



Now: why, instead of teleporting this state, we are not copying this state and send the copy to Bob? Well, in quantum mechanics, there is a "no-cloning" thing that doesn't allow us to do so. (Allan Adams!)
So, trying to do something else, Alice could measure the state, find α and β, and send the information to Bob. Wrong! Alice has just 1 "specimen" of the state, she can't copy it, so if she measures α and β she changes the state :-|
Third option: since Alice created the state, she could explain Bob how, and he could re-create: not possible, because α, β are real numbers, with ∞ figures: impossible to send :-|
Let's draw a diagram of what we're going to do! Let C be the state to be teleported:

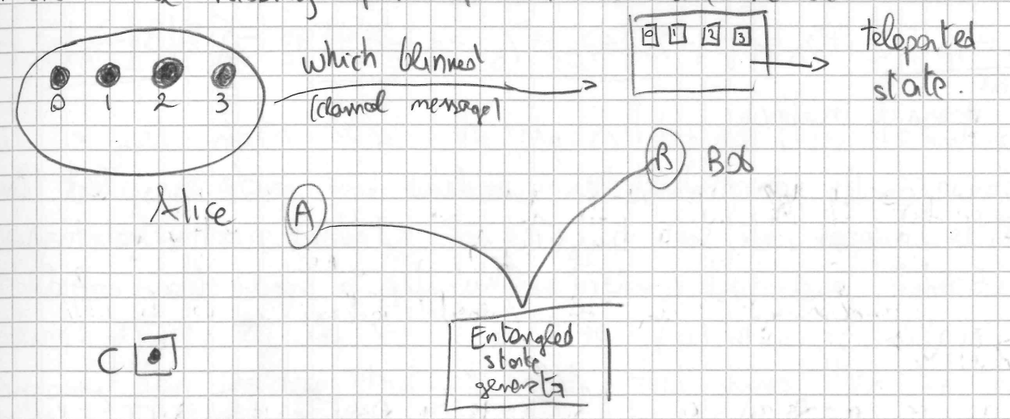
$$|\psi\rangle = \alpha|+\rangle_C + \beta|-\rangle_C .$$

The idea is to use, somehow, an entangled state. Basically,



Let us put an "entangle state source", states of 2 particles, and a particle B is given to Bob, particle A is given to Alice. A,B are an entangled pair.

Entanglement somehow correlates what goes in Bob with what goes in Alice. Entanglement happens instantaneously: we have no way of sending information through entanglement, since there is no way of learning what A does while B doesn't measure.
What is going to happen is: Alice performs a measurement between state A (entangled to B) and state C (the original one), and this is going to force the particle C to be teleported to Bob: B will become C.
There is a missing part of our scenario, re-drawn here:



Alice has 4 lights; after Alice will perform her measurement, one of the four lights will blink, and she will have to tell Bob which light blinked. This means sending to Bob the information in 2 bits: 4 lights, 2 bits required to label them. Bob will have a machine with 4 entries and, according to the information he gets, he will put the state B through the corresponding machine input.
Out, we claim, we will have the teleported state.

So. C is the state to be teleported, and (A,B) is an entangled state, so it will be one of the Bell states, like

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B\right)$$

Alice can handle only state A, Bob only B. Nevertheless, the state is entangled.
So, the total state comes by tensoring all the 3 particles:

$$|\Phi\rangle_{AB} \otimes \left(\alpha|+\rangle_C + \beta|-\rangle_C\right)$$

We're reaching the interesting point. Alice can act just on A, and C. So, Alice is going to perform a measurement by using a basis which includes just the two particles she has. So, Alice is going to use the Bell basis for A and C. So, $|\Psi_T\rangle =$

$= |\Phi_0\rangle_{AB} \otimes (\alpha|+\rangle_C + \beta|-\rangle_C) =$

$= \frac{1}{\sqrt{2}}(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B) \otimes (\alpha|+\rangle_C + \beta|-\rangle_C) =$

$= \frac{1}{\sqrt{2}}(\underbrace{\alpha|+\rangle_A|+\rangle_C}_{A,B,C}|+\rangle_B + \beta|+\rangle_A|-\rangle_C|+\rangle_B + \alpha|-\rangle_A|+\rangle_C|-\rangle_B + \beta|-\rangle_A|-\rangle_C|-\rangle_B)$

(the order doesn't matter) ↑

Ok. Now: we could use the states underlined here as a basis, but this would not be a good idea, since they are not entangled. Better go to the "entangled basis", and just re-write this $|\Psi_T\rangle$ in the basis of entangled states! So, with (4), we have:

$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|\Phi_0\rangle_{AC} + |\Phi_3\rangle_{AC}) \otimes (\alpha|+\rangle_B) + \frac{1}{2}(|\Phi_1\rangle_{AC} - i|\Phi_2\rangle_{AC}) \otimes (\beta|-\rangle_B) +$

$+ \frac{1}{2}(|\Phi_1\rangle_{AC} + i|\Phi_2\rangle_{AC}) \otimes (\alpha|-\rangle_B) + \frac{1}{2}(|\Phi_3\rangle_{AC} - |\Phi_0\rangle_{AC}) \otimes (\beta|-\rangle_B) =$

$=$ [we are doing nothing yet! Just rewriting stuff]

$= \frac{1}{2}|\Phi_0\rangle_{AC}\underbrace{(\alpha|+\rangle_B + \beta|-\rangle_B)}_{\text{This is exactly the state we want to teleport!}} + \frac{1}{2}|\Phi_1\rangle_{AC}(\beta|+\rangle_B + \alpha|-\rangle_B) +$

$+ \frac{1}{2}|\Phi_2\rangle_{AC}(i\alpha|-\rangle_B - i\beta|+\rangle_B) + \frac{1}{2}|\Phi_3\rangle(\alpha|+\rangle_B - \beta|-\rangle_B)$

This is the total state in the Bell basis. Now: since the state we are interested in, $|\psi\rangle_B$, is

$|\psi\rangle_B = \alpha|+\rangle_B + \beta|-\rangle_B$, [the one we want to teleport],

if we look carefully to what multiplies $|\Phi_3\rangle_{AC}$, we have that

$\alpha|+\rangle_B - \beta|-\rangle_B = \sigma_3|\psi\rangle_B$

The one multiplying $|\Phi_1\rangle_{AC}$ is $\sigma_1|\psi\rangle$, the one $|\Phi_2\rangle_{AC}$ is $\sigma_2|\psi\rangle$.

Good! So, we have

$|\Psi_T\rangle = \frac{1}{2}|\Phi_0\rangle_{AC}|\psi\rangle_B + \frac{1}{2}|\Phi_1\rangle_{AC}\sigma_1|\psi\rangle_B + \frac{1}{2}|\Phi_2\rangle_{AC}\sigma_2|\psi\rangle_B +$

$+ \frac{1}{2}|\Phi\rangle_{AC}\sigma_3|\psi\rangle_B$

This is just math. What's the physics behind these formulas? Well. Alice is going to measure in the Bell space AC. So, the four basis states are $|\Phi_0\rangle_{AC} \doteq |\Phi_3\rangle_{AC}$. As she measures, the state collapses into one of them. Then, the corresponding light starts to blink: 0, 1, 2, or 3.

Supposing for instance that the state zero shines, then the state collapses into $|\Phi_0\rangle_{AC}$. This new state has not any memory of the former state C, but B is sitted into the state we would like to teleport!!! So, Alice tells Bob to put state B into the machine "0" where there will be nothing: no magnetic field or other procedure! And, the same state goes out.

If light 1 blinks, Alice tells Bob to use the machine "1". This machine has a Hamiltonian realizing σ_1, through some magnetic field and time evolution, and again we get the right state!!!!

$\sigma_1(\sigma_1|\psi_B\rangle) = |\psi_B\rangle$!!!!

GOOD! This is the idea of teleportation. ;-)

Lecture 20: Multiparticle states and Tensor products (continued), and Angular Momentum

Now. We will look at the singlet states for a few properties and from Einstein-Podolsky-Rosen (EPR), and then Bell's answer.

We've been discussing this so-called singlet state:

$\frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2) = |\psi\rangle$ (1)

We calculated the z-component of total angular momentum (first and second particle), and it was 0. Also x- and y-: this state is rotationally-invariant!

This state is very interesting. In fact, this is also one entangled state, and it's not hard to realize physically. For instance, π_0 can decay in two photons, which have total angular momentum equal zero.

Or, more precisely, if we have the η_0 meson, which is one of the interacting particles involved in strong interactions, η_0 decays into a $\mu^+ + \mu^-$, which are two spin-$\frac{1}{2}$ particles. η_0 has 0 angular momentum, it's not spinning, and, if μ^+, μ^- goes into a state of 0 angular momentum, then conservation of angular momentum implies that μ^+, μ^- are in a state in the form of (1)!!

In homework it has been shown that the state can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\vec{n};+\rangle_1 |\vec{n};-\rangle_2 - |\vec{n};-\rangle_1 |\vec{n};+\rangle_2\right), \quad (2)$$

because it is invariant under rotation, and we could use, instead of the $|+\rangle, |-\rangle$ basis, any basis $\vec{n}_t$ related to any direction $\vec{n}$.

We're going to talk about probability, using symbols like $P(\vec{a}+, \vec{b}+)$, which is the probability to find the first particle with spin along $\vec{a}$ (so, $|\vec{a};+\rangle$), and the second particle with spin along $\vec{b}$ (so $|\vec{b};+\rangle$).

So, we're going to do some measurement and ask: what is the probability to find the first particle in a certain direction and another along another?

We're going to exploit (2) for this. So, if we write $|\psi\rangle$ in this form, we have:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|a+\rangle_1 |a-\rangle_2 - |a-\rangle_1 |a+\rangle_2\right),$$ because $\vec{n}$ can be anything, as just said!

Well, so, what is $P(\vec{a}+, \vec{b}+)$? We have to compute the "bra-ket" of our state! As usual. So:

$$P(\vec{a}+, \vec{b}+) = \left| \underbrace{\langle \vec{a};+ |}_{(1)} \otimes \underbrace{\langle \vec{b};+ |}_{(2)} \left(\frac{1}{\sqrt{2}}\left(|a+\rangle_1 |a-\rangle_2 - |a-\rangle_1 |a+\rangle_2\right)\right)\right|^2$$

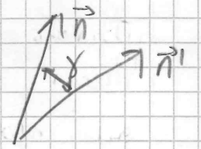
(1): the state for which we want probability

(2): Singlet

We know that $\langle a;+|$ with $|a;+\rangle$ gives 1, $\langle b;+|$ with $|a;-\rangle$ don't know, and so on: $\langle a;+|\, a;-\rangle = 0$, and we have (after these calculations):

$$P(\vec{a}+, \vec{b}+) = \frac{1}{2}\left|\langle b+ | a- \rangle\right|^2.$$

But we don't know how to compute it immediately! In this view, it is useful to remind that when we have any arbitrary 2-spin states with $\vec{n}$ and $\vec{n}'$, in homework 3/4, it was calculated the overlap of these two spin states, and it was:

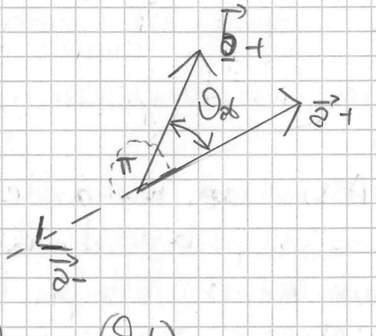


$$\left|\langle \vec{n} | \vec{n}' \rangle\right|^2 = \cos^2\frac{\gamma}{2}$$

So, here, we have:

$$P(\vec{a}+, \vec{b}+) = \frac{1}{2}\left|\langle \vec{b}+ | \vec{a}- \rangle\right|$$

Since one is $\vec{a}-$, one is $\vec{b}+$,



we have

$$P(\vec{a}+, \vec{b}+) = \frac{1}{2}\cos^2\left(\frac{\pi - \vartheta_{ab}}{2}\right) = \frac{1}{2}\sin^2\left(\frac{\vartheta_{ab}}{2}\right). \quad (3)$$

One more comment. If $\vec{b}$ was equal to $-\vec{a}$, then the would have maximum overlap! $\vartheta_{ab} = \pi$, $P(\vec{a}+, \vec{b}+) = \frac{1}{2}$. And this is $\frac{1}{2}$ because in half cases $|a\rangle$ is in $|a+\rangle$, in the other half it is $|a-\rangle$.

For instance, what if the first particle is in $|\vec{z}+\rangle$ and the second in $|\vec{x}+\rangle$? Which is the probability $P(\vec{z}+, \vec{x}+)$? Well, $\vec{x}$ and $\vec{z}$ form a $\frac{\pi}{2}$ angle, so we have:

$$P(\vec{z}+, \vec{x}+) = \frac{1}{2}\sin^2\left(\frac{\vartheta_{ab}}{2}\right) = \frac{1}{2}\sin^2\left(\frac{\pi}{4}\right) = \frac{1}{4}.$$

After this introduction, let's start with EPR

EPR

It all begins with a strange thing: "local realism". Something apparently pretty philosophical!

Local realism is given by basically two assumptions about measurement results:

① the measurement results should correspond to some aspect of reality. I mean: you measure something, you get a number, and this number then corresponds to some property of what we measured!

② the measurements that we do in the lab are not affected, whatsoever, by the measurements that somebody else is doing, at the same time, at the Moon.
I mean: there is no time for which the result obtained in a Moon lab has time to reach our Earth lab and somehow influence our measure. So: measurement is independent of actions performed at a distant location at the same time.

EPR, but specifically Einstein, claimed that physics must satisfy this. Well: these two things are not true! :-)

For ①. Assume we have a Stein-Gerlach [SG] apparatus, we put in some particle, a spin, and it goes up. We observe that it goes up, we then think that it had spin ↑. Meaning that Einstein claimed that it ALWAYS HAD spin up, that before the SG we just didn't know it!
So, if we put into the SG apparatus a bunch of spins, we admit that half of them have spin up the others spin down, and they move consequently.
But this is very different from the claim of quantum mechanics, for which the particle is in a superposition of spin up and down states, and after the measurement it collapses!
The way people try to correspond about this aspect of reality, is by introducing "hidden variables". So, the fact that there are some hidden things or properties, about the spin particle, that we don't know, but if we knew them then we would have known in advance the result of the SG experiment. [And it is possible to prove that this is not].

② is even more disturbing, because we are accustomed with the idea that locally, simultaneous things that cannot be reached, cannot interact via the exchange of light cannot affect each other.
And, even if reasonable, this is WRONG.

So, if ② is wrong: can information be sent at a speed faster than light? We are not going to discuss this but: even though things happening very far away can affect experiments, there is no real way of transmitting information faster than light!

We are going to discuss EPR. For ① we will not refer to hidden variables.
Let's discuss the first EPR experiment. We have two observers, Alice and Bob, located far away from each other. Alice is going to measure about the z-axis, Bob also; if we have a state like (1), if Alice measures spin up, Bob measures spin down:

$|+\rangle_1 |-\rangle_2$, and, if Alice measures down, Bob measures up:

$|-\rangle_1 |+\rangle_2$: THERE IS A CORRELATION!

EPR commented this saying that the pairs we've built, these so-called "entangled pairs", are pairs of particles with definite spins, spin directions! So, the EPR claim is that we've created pairs with two possibilities:

	Particle 1	Particle 2	
50%	$z+$	$z-$	(*)
50%	$z-$	$z+$	

So, not this "superposition hocus pocus". They are not correlated: they were already created with a spin or with the other. In this view, it's like having that the particle of Alice, A, has spin dependent on some λ, a variable which we don't know. If we would knew it, we know spin!

People say: ok, (*) is a simple situation! Let's add some measurements! Like, consider two directions! I mean, Alice could have two SG machines, one for $\vec{x}$ and one for $\vec{z}$, and also Bob!
So, if both A and B can measure in $\vec{z}$ OR $\vec{x}$, realism says that we don't know what we will get from measurements EVEN IF everything is already pre-determined: just we don't know the "hidden variables" ruling these quantities!

Let us label the particles such as

$(\hat{z}+;\hat{x}-)$

This is a particle such that if we measure its spin along z it would be a spin-up or if we would measure along x it then would a spin down. No entanglement: the particle has, in the claim of EPR, some attributes, ruled by the hidden variables. And we discover by measures, but they are already there, defined "a priori of the measurements".

Let us make a list of the possibilities:

EPR:	Particle 1	Particle 2
25%	$(\hat{z}+;\hat{x}+)$	$(\hat{z}-;\hat{x}-)$
25%	$(\hat{z}+;\hat{x}-)$	$(\hat{z}-;\hat{x}+)$
25%	$(\hat{z}-;\hat{x}+)$	$(\hat{z}+;\hat{x}-)$
25%.	$(\hat{z}-;\hat{x}-)$	$(\hat{z}+;\hat{x}+)$

So EPR claimed that 4 kinds of particles were generated, produced, each with 25% probability.

We could now ask questions to EPR. Like: what is the probability that

$P(z+ \text{ for } 1,\ z- \text{ for } 2)$? Well, it's just the first 2 cases, so 50% of the times!!! And this is also what would be predicted from the entangled states viewpoint!

Well, let's now ask something else!

$P(z+ \text{ on } 1,\ x+ \text{ on } 2)$? 25% of the time! Just like with QM!

And, then, it seems that EPR were right.

Then, Bell came, and tried with 3 directions!

Particles have then 3 attributes! 3 directions! So they will be labeled as

$(a+, b-, c+)$ for example. So if we measure

$\hat{S}\vec{a}$ we get $\frac{\hbar}{2}$, $\hat{S}\vec{b} = -\frac{\hbar}{2}$, $\hat{S}c = \frac{\hbar}{2}$!

(+) (−) (+)

Remark, again: not simultaneous measurements! Just one! The three according to EPR are three attributes!

Let's ask for probabilities, such as before.

EPR would do a list, just like before:

	Particle 1	Particle 2
N_1	$(a+;b+;c+)$	$(a-;b-;c-)$
N_2	$(a+;b+;c-)$	$(a-;b-;c+)$
N_3	$(a+;b-;c+)$	$(a-;b+;c-)$
N_4	$(a+;b-;c-)$	$(a-;b+;c+)$
N_5	$(a-;b+;c+)$	$(a+;b-;c-)$
N_6	$(a-;b+;c-)$	$(a+;b-;c+)$
N_7	$(a-;b-;c+)$	$(a+;b+;c-)$
N_8	$(a-;b-;c-)$	$(a+;b+;c+)$

Since particle 2 is correlated, it is "the opposite" for each row. Instead of putting $\frac{1}{8}$ to each population, let's say that each population has a different fraction: total N particles:

$N = \sum_{i=1}^{8} N_i$! (−)

Now, we're going to look for some contradiction in this! The formula (3) will contradict EPR!

Now, some questions. Which is the probability to get

$P(a+, b+)$? $= \frac{N_3+N_4}{N}$ (1° and 3° lines)

↑ particle 1, ↑ particle 2

$P(a+;c+) = \frac{N_2+N_4}{N}$

part 1, part 2

$P(c+;b+) = \frac{N_3+N_7}{N}$

part 1, part 2

Now: $\underline{N_3+N_4} \leq \underline{N_3} + N_7 + \underline{N_4} + N_2$

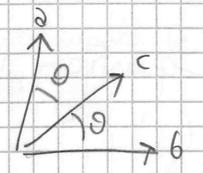
Whatever N_3 and N_2 are, certainly they are greater or equal than 0!
If we divide by N_1 we get

$$\frac{N_3+N_4}{N} \geq \frac{N_3+N_7}{N} + \frac{N_2+N_4}{N}$$

$$P(a+;b-) \geq P(a+;c+) + P(c+;b+) \qquad (4)$$

This is true for whatever population we choose!
This is Bell's inequality! This is a translation of the assumption of realism into an inequality!
But now, quantum mechanics has a formula for these probabilities!
So, if quantum mechanics is true, there might be some problem with this inequality! Otherwise, if this inequality was verified, we would have another possibility to have some hidden variable such that QM would be just a strange way of justifying things having a "classical" behaviour. If Bell's inequality doesn't hold, it means that QM does not represent "classical" stuff, but something new!
Using (3) we have

$$\cancel{\frac{1}{2}}\sin^2\left(\frac{\vartheta_{ab}}{2}\right) \geq \cancel{\frac{1}{2}}\sin^2\left(\frac{\vartheta_{ac}}{2}\right) + \cancel{\frac{1}{2}}\sin^2\left(\frac{\vartheta_{cb}}{2}\right)$$

So: (4) was derived from local realism. If realism is true, this should hold.
If we choose 3 vectors $\vec{a},\vec{b},\vec{c}$ like 

we have

$$\vartheta_{ab} = 2\vartheta \quad \vartheta_{ac} = \vartheta \quad \vartheta_{cb} = \vartheta$$

$$\sin^2(\vartheta) \geq \sin^2\left(\frac{\vartheta}{2}\right) + \sin^2\left(\frac{\vartheta}{2}\right)$$

For instance, if $\vartheta = \frac{\pi}{4}$, this is false!

This was a shock. This demonstrated that, in quantum mechanics, an experiment involving entangled particles leads to results that contradict (for some alignments of the experiment) the assumptions of local realism. Quantum mechanics is in contradiction with local realism. There's no way to keep the idea of quantum mechanics, but hidden variables, no effects of distance, and find no contradictions!

Angular momentum [1h 00]

Our purpose is to put together orbital and spin angular momenta.
We're going to use vector notations for operators and, in general, an "elegant" approach.
Simple things we know about orbital angular momentum:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$
$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad (*)$$
$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

First notation edit: instead of using the notation written above, let us use

$$\left(\hat{x} \quad \hat{y} \quad \hat{z}\right) \Longleftrightarrow \left(\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3\right)$$
$$\left(\hat{p}_x \quad \hat{p}_y \quad \hat{p}_z\right) \Longleftrightarrow \left(\hat{p}_1 \quad \hat{p}_2 \quad \hat{p}_3\right)$$

By this way, we can write the commutation relations in the form

$$\left[\hat{x}_i, \hat{p}_j\right] = i\hbar\,\delta_{ij}$$
$$\left[\hat{x}_i, \hat{x}_j\right] = \left[\hat{p}_i, \hat{p}_j\right] = 0$$

Also the angular momentum operators become easier, for example,

$$\hat{L}_1 = \hat{x}_2\hat{p}_3 - \hat{x}_3\hat{p}_2$$, without the complicated arrows!

When we talk about "vector notation" we have two chances: talk about triplets, or "form the vector themselves", which may sound disconcerting or at least abstract.
Let $\hat{r}$ be the vector operator

$$\hat{\vec{r}} = \hat{x}_1\vec{e}_1 + \hat{x}_2\vec{e}_2 + \hat{x}_3\vec{e}_3$$

Where the basis vectors are just useful to write expressions, but they don't really interact with the operators $\hat{x},\hat{y},\hat{z}$!

$$\hat{\vec{p}} = \hat{p}_1\vec{e}_1 + \hat{p}_2\vec{e}_2 + \hat{p}_3\vec{e}_3$$

And, finally,

$$\hat{\vec{L}} = \hat{L}_1\vec{e}_1 + \hat{L}_2\vec{e}_2 + \hat{L}_3\vec{e}_3$$, from (*)

We are basically treating operators as if they were numbers, even if obviously they are not! What changes with respect to the case of having numbers is: be careful of commutations!

Let $\vec{a}$, $\vec{b}$ be two of these "vector operators". The dot product between them should be:

$$\vec{a}\cdot\vec{b} = \hat{a}_i \hat{b}_i \qquad \text{(summed! we have repeated indexes!)}$$

But be careful of the ORDER!

Similarly,

$(\vec{a}\times\vec{b})_i$, the i-th component of $\vec{a}\times\vec{b}$, is defined as

$$\equiv \epsilon_{ijk}\, a_j\, b_k$$

And,

$$\vec{a}^2 = \vec{a}\cdot\vec{a} = a_i a_i \qquad \text{(definition)}$$

Now: is $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$? NO!

Let's calculate their difference!

$$\vec{a}\cdot\vec{b} - a_i b_i = \left[\hat{a}_i, \hat{b}_i\right] + b_i a_i \qquad \left(\text{since } [a_i, b_i] = a_i b_i - b_i a_i\right)$$

So,

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} + \left[\hat{a}_i, \hat{b}_i\right]$$

This is a new formula for operator vector analysis!

A very important corollary is that

$$\vec{r}\cdot\vec{p} = \vec{p}\cdot\vec{r} + \left[\hat{x}_i, \hat{p}_i\right] = \vec{p}\cdot\vec{r} + 3i\hbar$$

(because of the sum!)

Another nice formula is: the cross product is supposed to be antisymmetric. It is not! In fact,

$$(\vec{a}\times\vec{b})_i = \epsilon_{ijk}\,\hat{a}_i\,\hat{b}_k = \epsilon_{ijk}\left(\left[\hat{a}_j, \hat{b}_k\right] + \hat{b}_k\hat{a}_j\right) =$$

$$= \epsilon_{ijk}\left(\left[\hat{a}_j, \hat{b}_k\right]\right) - \epsilon_{ikj}\, b_k\, a_j =$$

$$= -\vec{b}\times\vec{a} + \epsilon_{ijk}\left[\hat{a}_j, \hat{b}_k\right] \quad !$$

This is the formula for the cross product! Not antisymmetric, due to the commutator!

So, how about $\vec{r}\times\vec{r}$? Is it 0?

$$\vec{r}\times\vec{r} = -\vec{r}\times\vec{r} + \epsilon_{ijk}\left[\hat{x}_j, \hat{x}_i\right] = 0 \quad !$$

$\vec{p}\times\vec{p} = 0$ (same reason!)

So, we would say that $\vec{L}\times\vec{L} = 0$... BUT IT NOT ZERO!

$\vec{L}\times\vec{L} \neq 0$!!!!

In fact,

$$\vec{L}\times\vec{L} = i\hbar\vec{L} \quad !$$

Then,

$$(\vec{r}\times\vec{p})_i = -\vec{p}\times\vec{r} + \epsilon_{ijk}\underbrace{\left[\hat{x}_j, \hat{p}_k\right]}_{i\hbar\,\delta_{jk}} = \ldots 0 \quad !$$

In fact, δ_{jk} is symmetric,

but ϵ_{ijk} is antisymmetric!

Let's see:

$\epsilon_{ijk}\,\delta_{jk}$ = relabel j and k

$= \epsilon_{ikj}\,\delta_{kj} = -\epsilon_{ijk}\,\delta_{jk}$ (when I switch labels I must change sign of ϵ_{ijk}!)

So, the expression is equal to "minus itself" and then it's 0! ;-)

So, this is what we call $\vec{L}$:

$$\vec{L} = \vec{r}\times\vec{p} \quad ! \qquad \left(= -\vec{p}\times\vec{r} \quad !\right)$$

Another relation

$$\vec{r}\cdot\vec{L} = \vec{r}\cdot(\vec{r}\times\vec{p}) = \hat{x}_i\,\epsilon_{ijk}\,\hat{x}_j\,\hat{p}_k = \epsilon_{ijk}\,\hat{x}_i\,\hat{x}_j\,\hat{p}_k,$$

but since $\hat{x}_i\,\hat{x}_j$ commute, they are symmetric in i and j, and with the antisymmetric ϵ_{ijk} we have 0!

So $\vec{r} \cdot \vec{L} = 0$.

What about $\vec{p} \cdot \vec{L}$?

$\vec{p} \cdot \vec{L} = \hat{p}_i \, \epsilon_{ijk} \, \hat{x}_j \, \hat{p}_k$

It is not immediately possible to exploit symmetry, since we have the $\hat{x}_j$ in the middle!

$= \epsilon_{ijk} \, \hat{p}_i \, \hat{x}_j \, \hat{p}_k$

But $\hat{p}_i \hat{x}_j$ is δ_{ij} and thus vanishes against the ϵ_{ijk}! So, we get

$= \epsilon_{ijk} \, x_j \, \hat{p}_i \hat{p}_k$ and $\hat{p}_i \hat{p}_k$ is symmetric in i and k!

$= 0$!

Alternative proof: $\vec{L} = \vec{r} \times \vec{p} = -\vec{p} \times \vec{r}$ (-)